ELEMENTARY CALCULUS

SMITH

ELEMENTARY CALCULUS

A TEXT-BOOK FOR THE USE OF
STUDENTS IN GENERAL SCIENCE

BY

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PREFACE

THIS volume has been written in response to the unmistakable and growing demand for a text-book on the Calculus which shall present in a course of from thirty-five to forty exercises the fundamental notions of this branch of mathematics. In American technical schools students pursuing courses distinct from engineering branches usually terminate their mathematical studies with Plane Analytic Geometry. But in view of the recent remarkable development of certain of the general sciences along mathematical lines, such a course can no longer be regarded as adequate. Moreover, there can be no difference of opinion as to the relative advantage to the student of a knowledge of more than the mere elements of Analytic Geometry and an introductory acquaintance with the Calculus. It is, I think, the experience of every teacher that the average student first realizes the power and use of mathematics when taught to solve problems in maxima and minima by means of the methods of the Differential Calculus. Certainly no stronger argument can be adduced in favor of an adjustment of the curriculum which shall include this branch of mathematics. Such a change has been effected in the Sheffield Scientific School, and results abundantly justify the step.

For the general student in our colleges who elects a year's work in mathematics beyond the usually required

Trigonometry, the most satisfactory course would seem to be one in which the time is equally divided between Plane Analytic Geometry and Calculus.

In writing this book I have everywhere emphasized the possibility of applications. The examples have been carefully selected with this end in view. The first chapter may seem long, but the notion of limit certainly demands adequate treatment. While an elementary text-book offers no excuse for employment of the refinements of modern rigor, I have endeavored to avoid positive inaccuracies and have carefully distinguished between demonstration and illustration.

I am indebted to my colleague, Dr. W. A. Granville, for many helpful suggestions.

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CONTENTS

ELEMENTARY CALCULUS

CHAPTER I

FUNCTIONS AND LIMITS

1. Continuous Variation. In this book we are concerned with *real numbers* only. Geometrically, such numbers may be conveniently represented by points of a scale (Fig. 1).

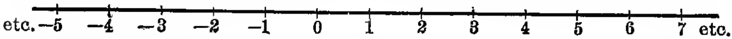


FIG. 1

Then to every real number corresponds one point of the scale, and only one; conversely, every point of the scale represents a real number. Any segment of the scale, however small, represents indefinitely many numbers. We speak indifferently of the number a and the point a of the scale.

A variable x is said to *vary continuously* between the numbers a and b when it assumes values corresponding to *every* point of the segment ab.

2. Functions. The problems arising in Elementary Calculus involve in general two variables in such a way that the value of one variable can be calculated as soon as a value is assumed for the other. Thus, in Geometry, the student has an illustration in the area and radius of a circle, two variables such that the area A can be calculated when we know the radius r from the formula $A = \pi r^2$.

Definition. *A variable is said to be a function of a second variable when its value depends upon the value of the second variable and can be calculated when the value of the second variable is assumed.*

The first variable is called the *dependent variable*, and the second the *independent variable.*

For example, the equations

$$y = x^2, \quad y = \sin x, \quad y = \log_{10}(x^2 - 1)$$

state that y is a function of x. In the first two cases, y may be calculated for *any* value of x; in the last case, however, x is restricted to values *numerically greater than* 1, since the logarithms of negative numbers cannot be found. In the first two cases, then, we say that the dependent variable (or the function) is *defined* for every value of x, and in the last the function is *defined* only when x exceeds 1 numerically.

A function is **defined** *for a value of the variable when its value can be calculated for that value of the variable.*

Elementary Functions. *Power Function:* x^m, m any *positive integer.*

Logarithmic Function: $\log_a x$, $a > 0$; this function is defined only for $x > 0$.

Exponential Function: a^x, $a > 0$, *i.e.* the exponent is a variable, the number a being a constant.

*Circular Functions:** $\sin x$, $\cos x$, $\tan x$, etc., *i.e.* involving the six trigonometric functions.

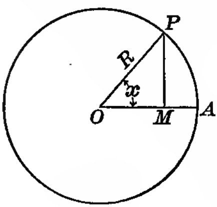


FIG. 2

* So called from the use of the circle in their definition, *e.g.* in Fig. 2, $\sin AOP = \frac{MP}{R} = MP$ if R is the *unit* of linear measure. Hereafter, angles will always be measured in *circular measure*, *i.e.* $x = \frac{\text{arc}}{\text{radius}} = \frac{AP}{R}$. In the *unit circle*, $R = 1$, $x = \text{arc}\, AP$.

Inverse Circular Functions: arc sin x, arc tan x, etc., *i.e.* the "arc whose sine is x," "arc whose tangent is x," etc. In the unit circle (see Fig. 2) $R = 1$; if $x = MP$, then arc sin $x =$ arc AP.

One thing is peculiar here. Assuming any value of x *not exceeding* 1 *numerically*, arc sin x may be calculated, but the number of answers is always *indefinitely great.* For not only is

$$\text{arc sin } MP = \text{arc } AP,$$

but also equal to any number of circumferences $+$ arc AP,

i.e. $$\text{arc sin } MP = \text{arc } AP + 2\pi n,$$

where n is any integer.

For this reason the inverse circular functions are called *many-valued* functions. For definiteness we may always take the *least* positive arc.

3. Functional Notation. As general symbols for functions of variables we use the notation

$$f(x),\ \theta(y),\ \phi(r),\ \text{etc.},$$

(read f function of x, theta function of y, phi function of r, etc.).

We mean by this that $f(x)$ is a variable whose value depends upon x, and can be found when a value is assumed for x. The notation is extremely convenient, for it enables us to indicate the *value of the function* corresponding to any value of the variable for which the function is defined.

Thus $f(a)$ represents the value of $f(x)$ for $x = a$, $\theta(0)$ the value of $\theta(y)$ for $y = 0$, $\phi(\frac{1}{2})$ the value of $\phi(r)$ for $r = \frac{1}{2}$, etc.

EXERCISE 1

1. For what values of the variable are the following functions defined?

(*a*) $\frac{1}{x}$. *Ans.* For every value except $x = 0$, since $\frac{1}{0}$ cannot be calculated.*

(*b*) $\sqrt{x^2 - 6x}$. Since $x^2 - 6x$ or $x(x-6)$ must not be *negative*, x and $x - 6$ must always have the same signs.

Ans. For every value except those between 0 and 6.

(*c*) $\sqrt{y - y^2}$; (*d*) $\sqrt[x]{10}$; (*e*) $\arcsin x$;

(*f*) $\operatorname{arc\,sec} x$; (*g*) $\sin\sqrt{1+x}$; (*h*) $\log\tan x$.

2. Given $f(x) \equiv x^3 - 7x^2 + 16x - 12$, show that $f(2) = 0$, $f(3) = 0$. Does $f(x)$ vanish for any other value of x?

3. Given $f(x) = \log x$; show that

$$f(x) + f(y) = f(xy).$$

4. Given $\phi(x) = a^x$; show that

$$\phi(x)\,\phi(y) = \phi(x+y).$$

5. Given $\theta(x) = \cos x$;

then $$\theta(x) + \theta(y) = \cos x + \cos y.$$

From Trigonometry, we know that

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y);$$

$$\therefore\ \theta(x) + \theta(y) = 2\,\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{x-y}{2}\right).$$

4. Graph of a Function. After determining for what values of the variable a given function is defined, it is important to know in what manner the value of the function

* The student should observe that the four fundamental operations of arithmetic, addition, subtraction, multiplication, and division, when performed with *real* numbers, give *real* numbers, with the single exception that *division by zero is excluded.*

changes with the variable. Geometrically this is accomplished by drawing the *graph of the function*, which is thus defined:

The graph of a function is the curve passing through all points whose abscissas are the values of the variable and ordinates the corresponding values of the function.

In the language of Analytic Geometry the graph of a function $f(x)$ is the *locus of the equation*

$$y = f(x).$$

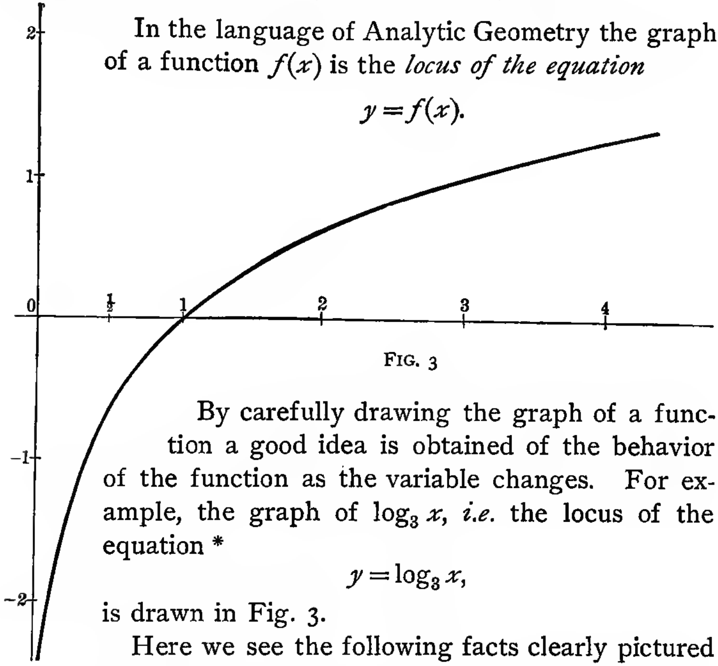


FIG. 3

By carefully drawing the graph of a function a good idea is obtained of the behavior of the function as the variable changes. For example, the graph of $\log_3 x$, *i.e.* the locus of the equation *

$$y = \log_3 x,$$

is drawn in Fig. 3.

Here we see the following facts clearly pictured to the eye.

(*a*) For $x = 1$, $\log_3 x = \log_3 1 = 0$.

(*b*) For $x > 1$, $\log_3 x$ is positive and increases as x increases.

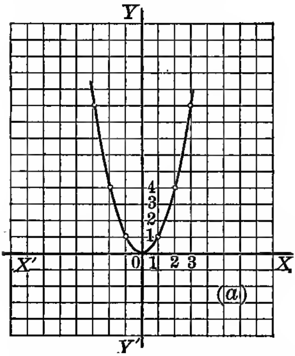
* The values of y are found from the formula proven in the theory of logarithms, $\log_3 x = \dfrac{\log_{10} x}{\log_{10} 3}$.

(*c*) For $x < 1$, $\log_3 x$ is negative and increases indefinitely in *numerical value* as x diminishes.

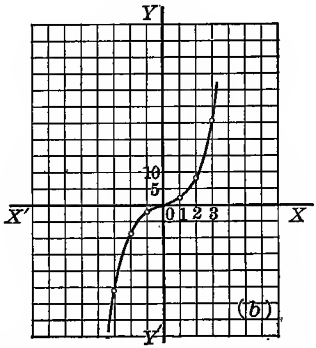
(*d*) For $x = 0$, $\log_3 x$ is not defined, since the logarithm of zero cannot be calculated.

The graph of the general logarithmic function $\log_a x$ may be drawn by merely changing the ordinates in Fig. 3 in the constant ratio $1 \div \log_3 a$.

Graphs: (*a*) Of x^2.

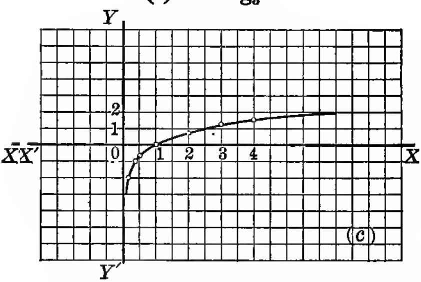


(*b*) Of x^3.

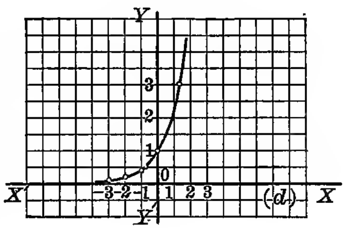


The graph of x^m has the appearance of (*a*) or (*b*) according as m is even or odd.

(*c*) Of $\log_3 x$.

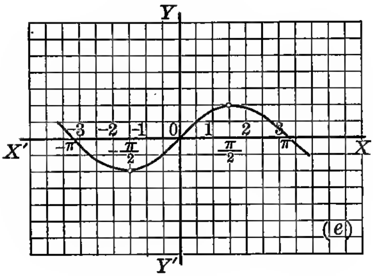


(*d*) Of 3^x.

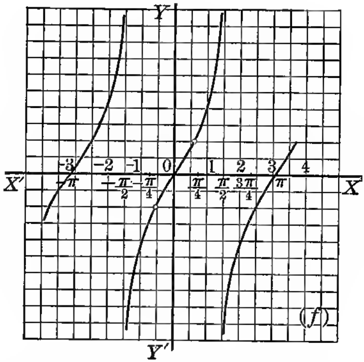


Since if we set $y = \log_3 x$, then $x = 3^y$, the graph in (*c*) has the same relation to XX' and YY' as (*d*) to YY' and XX'.

(*e*) $\sin x$.



(*f*) $\tan x$.



The graphs of the circular functions have the appearance of a curve repeated over and over as the variable increases or decreases. As in (*c*) and (*d*), if we revolve (*e*) and (*f*) around XX', and interchange XX' and YY', we shall have the graphs of $\operatorname{arc}\sin x$ and $\operatorname{arc}\tan x$ respectively.

5. Limits. For the study of the Calculus it is absolutely essential that the student should understand perfectly the fundamental notion of a limit. He is already familiar with simple examples of limits from Geometry, such as the limit of the perimeter of an inscribed regular polygon as the number of sides is indefinitely increased is the circumference, and the limit of the area of the polygon is the area of the circle. These are examples of *variables approaching limits*, the variable being in the first case the perimeter, and in the second the area of the regular polygon. The following definition states the matter generally.

Definition. *A variable is said to approach a number A as a limit when the values of the variable ultimately differ*

from A by a number whose numerical value is less than any assignable positive number.

If we represent the values of the variable by the infinite sequence

$$a_1,\ a_2,\ a_3,\ \cdots,\quad a_n,\ a_{n+1},\ \cdots,$$

then on the scale (Fig. 1) the points corresponding to a_1, a_2, a_3, $\cdots$, a_n, a_{n+1}, $\cdots$, etc., will ultimately approach nearer the point A than any assignable length, that is, will "heap up" at the point A. The definition interpreted

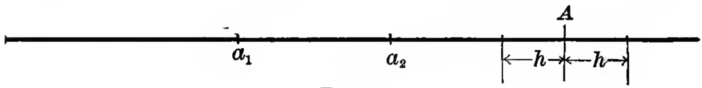


FIG. 4

geometrically means, then, that *no length h* (Fig. 4), *however small*, can be laid off from the point A, but that points of the sequence will fall within the segment.

We write Limit $(a_n) = A$, or, also, if we denote the variable whose values are a_1, a_2, etc., by x,

$$\text{Limit}\,(x) = A.$$

6. Limiting Value of a Function. Continuous Function. Consider the elementary function $\log_a x$ (Fig. 3). Take any sequence

$$a_1,\ a_2,\ a_3,\ \cdots,$$

of positive numbers whose limit is some positive number A. For example, the sequence

$$1.3,\ 1.33,\ 1.333,\ \cdots,$$

the limit of which is $\frac{4}{3}$. Consider now the sequence of numbers

$$\log_a(a_1),\ \log_a(a_2),\ \log_a(a_3),\ \cdots,$$

and draw their ordinates in Fig. 3. Then the student

will see that this last sequence has the limit $\log_a(A)$; that is,

When the variable x approaches a limit A greater than zero, the logarithmic function $\log_a x$ approaches the limit $\log_a A$.

We express this important fact by writing

$$\text{Limit}\,(\log_a x)_{x=A} = \log_a A.$$

The general relation brought out by this example is the following: When the values assumed by the variable x approach * a limiting value A, then the corresponding values of the function will also approach a limiting value; and if the function is defined for the value A, then the limiting value of the function is its value for $x = A$. Or, in symbols, if $f(A)$ is a number, then

$$\text{Limit}\,(f(x))_{x=A} = f(A).$$

For example, since $\cos 0 = 1$, $\text{Limit}\,(\cos x)_{x=0} = 1$. The property above described is that of *continuity*; *i.e.* a *continuous function* is such that

$$\text{Limit}\, f(x) = f(\text{Limit}\, x).$$

For the purposes of the Calculus it is essential that a function should be continuous. The elementary functions of § 2 possess this property.

7. Infinity. If the points on the scale of Fig. 1 corresponding to the sequence of values of the variable x

* The variable x may approach the limit A in *any* manner consistent with the definition of the function. In the above illustration the geometrical sequence

$$1,\ 1+\tfrac{1}{4},\ 1+\tfrac{1}{4}+\tfrac{1}{16},\ 1+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}, \text{ etc.,}$$

whose limit is $\frac{4}{3}$, might also have been taken.

ultimately advance to the right without limit, we say, "x increases without limit," or also, "x approaches the limit positive infinity," and we write

$$\text{Limit } x = +\infty.$$

If under the same conditions the points advance to the left without limit, we say, "x decreases without limit," and write

$$\text{Limit } x = -\infty.$$

Finally, if the points advance both to the right and left without limit, we write

$$\text{Limit } x = \infty.$$

The student should disabuse his mind of any previous notions of infinity not agreeing with the above definitions. The symbols $+\infty$, $-\infty$, ∞, must be used always in the sense above described.

8. Fundamental Theorems on Limits. The student is asked to accept the following theorems as true:

Given a number of variables whose limits are known; then

I. *The limit of an algebraic sum of any finite number of variables equals the same algebraic sum of their respective limits.*

II. *The limit of the product of any finite number of variables equals the product of their respective limits.*

III. *The limit of a quotient of two variables equals the quotient of their respective limits when the limit of the denominator is not zero.*

9. Two Important Limits. To prove *

$$\textbf{Limit}\left[\frac{\sin x}{x}\right]_{x=0}=1.$$

In Fig. 5 let $x = \text{arc } AT = \text{arc } AS$, the radius OQ being taken equal to unity. Then

$$\sin x = MT = SM,$$

$$\tan x = TQ = QS.$$

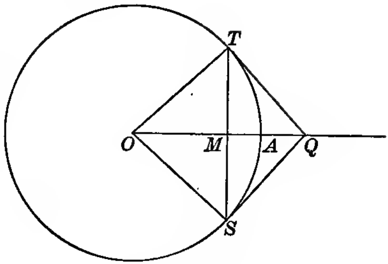


FIG. 5

Now $$ST < \text{arc } ST < SQ + QT.$$

$$\therefore\ 2\sin x < 2x < 2\tan x;$$

whence, dividing through by $2 \sin x$,

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}\left(=\frac{1}{\cos x}\right).$$

Therefore, taking reciprocals,

$$\cos x < \frac{\sin x}{x} < 1.$$

Now let x approach zero as a limit; then, since $\cos 0 = 1$, and the value of $\dfrac{\sin x}{x}$ lies between 1 and $\cos x$, we must have

$$\text{Limit}\left[\frac{\sin x}{x}\right]_{x=0} = 1.$$

10. Consider next the infinite sequence

$$1,\ 1+\frac{1}{1},\ 1+\frac{1}{1\cdot 2},\ 1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3},\ \cdots.$$

* Since for $x = 0$, $\dfrac{\sin x}{x} = \dfrac{0}{0}$, a meaningless expression, the function $\dfrac{\sin x}{x}$ is not defined for $x = 0$.

Representing the successive terms by a_1, a_2, a_3, $\cdots$, we have

$$\begin{aligned}
a_1 &= 1, && = 1 \\
a_2 &= 1 + \frac{1}{1}, && = 2 \\
a_3 &= 1 + \frac{1}{1} + \frac{1}{1\cdot 2}, && = 2.5 \\
a_4 &= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3}, && = 2.666\cdots \\
a_n &= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{\underline{|n-1}}\,^{*}, \text{ etc.}
\end{aligned}$$

The numbers of this sequence continually increase. We may show, however, that any term is less than 3.

For $\underline{|r} > 2^r$, and therefore

$$a_n < 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1},$$

since
$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

is a geometrical progression and its sum may be immediately written by the usual formula.

Hence $a_n < 3 - \frac{1}{2^{n-1}}$, and taking $n = 1, 2, 3$, etc. *ad infinitum*, every term of the sequence is seen to be less than 3.

FIG. 6

The points, then, corresponding to the sequence (Fig. 6) must heap up at some point to the left of 3; that is, the sequence must have a limit.

* The symbol $\underline{|n-1}$, read "factorial $n-1$," means the product of all integers from 1 to $n-1$ inclusive.

The calculation of this limit to any number of decimal places is a matter of no difficulty, as the following computation to five decimal places will show.

$$
\begin{array}{lrl}
\text{Write down} & 1.000000 & (= 1). \\[2ex]
\text{Divide by} & 2\,)\,1.000000 & \left(=\dfrac{1}{\underline{|1}}\right) \\[2ex]
 & 3\,).500000 & \left(=\dfrac{1}{\underline{|2}}\right) \\[2ex]
 & 4\,).166667 & \left(=\dfrac{1}{\underline{|3}}\right) \\[2ex]
 & 5\,).041667 & \left(=\dfrac{1}{\underline{|4}}\right) \\[2ex]
 & 6\,).008333 & \left(=\dfrac{1}{\underline{|5}}\right) \\[2ex]
 & 7\,).001388 & \left(=\dfrac{1}{\underline{|6}}\right) \\[2ex]
 & 8\,).000198 & \left(=\dfrac{1}{\underline{|7}}\right) \\[2ex]
 & 9\,).000025 & \left(=\dfrac{1}{\underline{|8}}\right) \\[2ex]
 & 10\,).000003 & \left(=\dfrac{1}{\underline{|9}}\right) \\
\hline
\text{Adding,} & 2.71828 &
\end{array}
$$

neglecting the figure in the sixth decimal place, of which we cannot be sure. In fact, it can be easily shown that

2.71828 is the limit of the sequence correct to five decimal places.

Writing the limit of the sequence in the form of an infinite series and denoting this limit by e, we have

$$e = 1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \text{etc., } \textit{ad infinitum}.$$

$$e = 2.71828 \cdots.$$

The number e is called in the Theory of Logarithms the *Napierian base* or *natural base*, and is a number of prime importance in mathematics.

The expression for e in the form of an infinite series should be remembered and also its value to five decimal places.

11. To prove

$$\text{Limit}\left[1 + \frac{1}{z}\right]^{z}_{z=\infty} = e.$$

A rigorous proof of this very important limit is beyond the scope of this volume. We may perhaps best illustrate the meaning of the theorem by drawing the graph of the function for positive values of z.

Setting $\qquad y = \left(1 + \frac{1}{z}\right)^{z}$, then

$$\log_{10} y = z \log_{10}\left(1 + \frac{1}{z}\right),$$

and for any value of z greater than zero y may be approximately calculated, as for example in the accompanying table, which gives y to five decimal places.

z	y
.01	1.04723
.1	1.27098
1.	2.
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

etc.

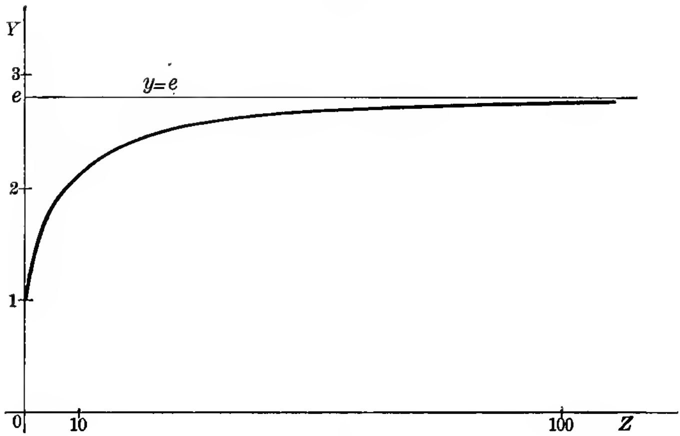


FIG. 7

The figure illustrates the theorem in showing that the graph approaches the line $y = e$ as z increases indefinitely. When z diminishes toward zero, y approaches unity.

EXERCISE 2

[*The graph of the function considered must be drawn in every case.*]

1. Prove $$\operatorname{Limit}\left[\frac{x^2-3x+4}{x-1}\right]_{x=2}=2.$$

We have merely to substitute 2 for x.

2. Prove $$\operatorname{Limit}\left[\frac{x^2-a^2}{x+a}\right]_{x=-a}=-2a.$$

We cannot substitute directly, for we should get $\frac{0}{0}$, a meaningless expression. But $\frac{x^2-a^2}{x+a}=x-a$, and we may now substitute.

3. Prove the following in which a is any number greater than zero:

$$\operatorname{Limit}\left[\frac{a}{x^2}\right]_{x=0}=+\infty;\quad \operatorname{Limit}\left[\frac{a}{x}\right]_{x=0}=\infty;$$

$$\operatorname{Limit}\left[ax\right]_{x=\infty}=\infty;\quad \operatorname{Limit}\left[\frac{a}{x}\right]_{x=\infty}=0.$$

The last three results are often written

$$\frac{a}{0}=\infty,\quad a\cdot\infty=\infty,\quad \frac{a}{\infty}=0,$$

but the student must remember that such equations are merely abbreviations of the preceding.

4. Prove $$\operatorname{Limit}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right]_{h=0}=\frac{1}{2\sqrt{x}}.$$

HINT. Multiply numerator and denominator by $\sqrt{x+h}+\sqrt{x}$.

5. Show that

$$\operatorname{Limit}\left[\frac{\tan x}{\sin x}\right]_{x=0}=1;\quad \operatorname{Limit}\left[\tan x\right]_{x=\frac{\pi}{2}}=\infty;$$

$$\operatorname{Limit}\left[\log_e x\right]_{x=0}=-\infty;\quad \operatorname{Limit}\left[e^{-x}\right]_{x=\infty}=0.$$

CHAPTER II

DIFFERENTIATION

12. Increments. In order to understand the manner of variation of a function as the variable varies, it is essential to know how great a change in value occurs in the function for a given change in value of the variable. Change in value is termed *increment; i.e. the increment of the function is the change in value of the function corresponding to a given change in value or increment of the variable.*

The problem now arises: *To calculate the increment of a given function.*

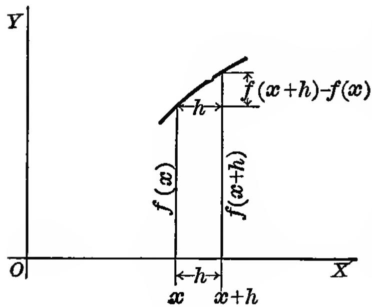


FIG. 8

Let $f(x)$ be defined for all values of x from x to $x+h$ (Fig. 8). Now for $x+h$ the value of the function is $f(x+h)$, hence the *increment of the function $f(x)$ corresponding to an increment h in the variable x is*

$$f(x+h)-f(x).$$

We shall represent the increment of any variable by the letter Δ (read "delta") prefixed to that variable, thus

$$\textbf{If}\quad \Delta x = h,\quad \textbf{then}\quad \Delta f(x) = f(x+h) - f(x).$$

RULE. *To find the increment of a function, calculate the new value of the function by replacing x by $x + h$ and subtract the old value of the function from the new value.*

EXERCISE 3

1. Find Δx^2. $\qquad \Delta x^2 = (x+h)^2 - x^2 = 2hx + h^2$. *Ans.*

2. Find $\Delta\left(\frac{1}{x}\right)$. $\qquad \Delta\left(\frac{1}{x}\right) = \frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$. *Ans.*

3. Prove $\Delta\sqrt{x} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$. (Ex. 4, page 22.)

4. Find $\Delta \log x$.

$$\Delta \log x = \log(x+h) - \log x = \log\left(\frac{x+h}{x}\right) = \log\left(1 + \frac{h}{x}\right). \quad \textit{Ans.}$$

5. Find $\Delta \sin x$.

$$\Delta \sin x = \sin(x+h) - \sin x = 2\cos(x + \tfrac{1}{2}h)\sin\tfrac{1}{2}h,$$

from Trigonometry. *Ans.*

6. Find Δe^x. $\qquad \Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$. *Ans.*

7. Find $\Delta \cos 2x$. $\qquad \Delta \cos 2x = -2\sin(2x+h)\sin h$. *Ans.*

8. Find $\Delta\sqrt{1+x}$. $\qquad \frac{h}{2\sqrt{1+x}}$. *Ans.*

13. The Increment Quotient. While the increment of a function as found in the preceding article is of importance, still more essential in any investigation is the *rate of change of the function*, that is, *the change in the function per unit change in the variable.*

If we form the quotient

$$\frac{\Delta f(x)}{\Delta x},$$

we obtain the *average* rate of change of the function while the variable changes from x to $x + h$.

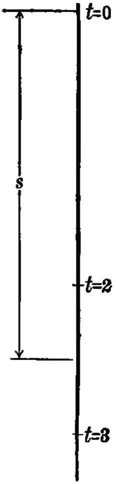


FIG. 9

For example, the "law of falling bodies," given in Mechanics, asserts that the distance s traversed by such a body falling freely from rest in a vacuum varies as the square of the time t, that is,

$$s = 16.1\, t^2,$$

the constant 16.1 being determined experimentally when s is measured in feet and t in seconds.

Therefore $\Delta s = 16.1\,(t + h)^2 - 16.1\, t^2$,

or $\quad \dfrac{\Delta s}{\Delta t} = 16.1\,(2\,t + h)$, since $\Delta t = h$.

For example, the *average* velocity *throughout the third second* is given by setting in $\dfrac{\Delta s}{\Delta t}$, $t = 2$, $h = 1$, and is 80.5 feet per second.

EXAMPLES

1. From Physics we learn that for a perfect gas at constant temperature the product of the pressure p and volume v is constant, or $pv =$ a constant c, i.e. $p = \dfrac{c}{v}$; show that $\dfrac{\Delta p}{\Delta v} = -\dfrac{c}{v^2 + v\Delta v}$.

2. Show from Ohm's law, viz. current strength C equals electromotive force E divided by the resistance R, that for constant R the change of current strength per unit change of electromotive force is constantly equal to $1 \div R$.

14. Derivative of a Function. In the illustration taken from the law of falling bodies given in § 13, let us propose to ourselves to find *the velocity at the end of two seconds.* Making $t = 2$, we have

$$\frac{\Delta s}{\Delta t} = 64.4 + 16.1\, h,$$

which gives us the *average velocity* throughout any time h after two seconds of falling. Our notion of velocity shows us, however, that by the velocity *at the end of two seconds* we do *not* mean the *average* velocity during one second after that moment, or even during $\frac{1}{100}$ or $\frac{1}{1000}$ of a second after that moment, but, in fact, we mean the *limit* of the average velocity when h diminishes toward zero; that is, the velocity at the end of two seconds is 64.4 feet per second. Thus, even the everyday notion of velocity involves mathematically the notion of a limit, or, in our notation,

$$\textbf{Velocity} = \textbf{Limit}\left[\frac{\Delta s}{\Delta t}\right]_{\Delta t=0}.$$

Thus, after t seconds have elapsed, the velocity of a falling body is $32.2\,t$ feet per second.

Again, let it be required to find the slope of the tangent at any point P of a plane curve whose equation is given in rectangular coördinates x and y.

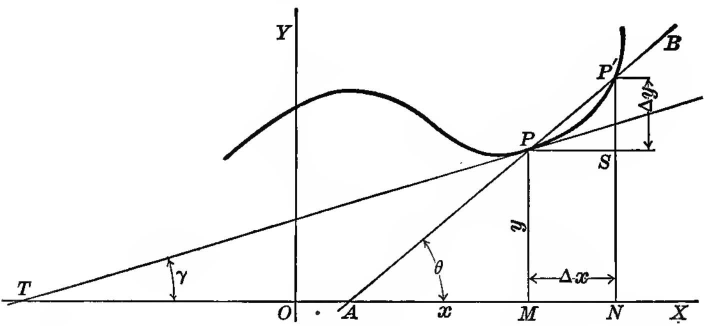


FIG. 10

The tangent at P is constructed as follows (Fig. 10):

Through P and any point P' on the curve near P draw the secant AB. Let the point P' move along the curve

toward P, the secant AB meanwhile turning around P. Then when P' coincides with P, the secant AB becomes the tangent TP.

Now if P is (x, y) and P' $(x+\Delta x, y+\Delta y)$, the *slope* of AB is

$$\tan\theta = \frac{SP'}{PS} = \frac{\Delta y}{\Delta x}.$$

As P' approaches P as above described, Δx will approach zero as a limit, while θ approaches the angle PTO or γ; hence, at the limit,

$$\tan\gamma = \text{Limit}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x=0} = \textbf{slope of tangent at any point } \boldsymbol{P}.$$

For example, the slope of the tangent at any point of the parabola $y = x^2 + 3$ is $2x$.

Law of Linear Expansion. If l_0 is the length of a rod at 0° Centigrade, and l the length at $t°$ on the same scale, then experiment establishes the law of expansion

$$l = l_0 + at + bt^2,$$

a and b being constants. The *coefficient of linear expansion* at any temperature t is the increase in length per unit change in temperature, *i.e.*

$$\textit{coefficient of expansion} = \text{Limit}\left(\frac{\Delta l}{\Delta t}\right)_{\Delta t=0}.$$

We easily find, then, that

$$\textit{coefficient of expansion} = a + 2\,bt,$$

and $\quad\therefore\ a =$ the coefficient of expansion at 0°.

Specific Heat of a Substance. The specific heat of any substance is the quantity of heat necessary to raise a unit mass of the substance one degree in temperature. If Q is the measure of the quantity of heat in unit mass, and t the corresponding temperature, then by definition,

$$\textit{specific heat} = \text{Limit}\left(\frac{\Delta Q}{\Delta t}\right)_{\Delta t=0}.$$

These examples show that we obtain an important *new* function of the variable if we can find the limit of the Increment Quotient when the increment of the variable approaches zero. This function is called the **derivative** of the function.

DEFINITION. **The derivative of a function is the limit of the quotient of the increment of the function and the increment of the variable when the latter increment approaches the limit zero.**

The step of finding the limit of $\frac{\Delta f(x)}{\Delta x}$ when Δx approaches 0 is indicated by changing the Δ's to ordinary d's, so that $\frac{df(x)}{dx} = \text{Limit}\left(\frac{\Delta f(x)}{\Delta x}\right)_{\Delta x=0}$, or also, if

$$\Delta x = h, \quad \frac{df(x)}{dx} = \text{Limit}\left[\frac{f(x+h)-f(x)}{h}\right]_{h=0}.$$

The symbol $\frac{df(x)}{dx}$ is read, "derivative of $f(x)$ with respect to x." This, being a new function of x, is often written $f'(x)$, so that also,

$$\frac{df(x)}{dx} = f'(x).$$

Thus in the illustrations given,

$$\text{velocity} = \frac{ds}{dt},$$

i.e. *velocity is the derivative of the space traversed in the time t with respect to the time.*

$$\text{Slope of tangent} = \frac{dy}{dx},$$

i.e. *equals the derivative of the ordinate of the point with respect to its abscissa.*

$$\text{Coefficient of linear expansion} = \frac{dl}{dt},$$

or *the derivative of the length with respect to the temperature.*

$$\text{Specific heat} = \frac{dQ}{dt},$$

that is, *equals the derivative of the quantity of heat in unit mass with respect to the temperature.*

Many more illustrations of physical magnitudes might be given which take the form of a derivative.

We call $\frac{d}{dx}$ the *sign of differentiation*, so that the prefixing of $\frac{d}{dx}$ to any function of x means that the following process is to be carried through:

General Rule of Differentiation. **1°. Calculate the quotient of the increment of the function and the increment of the variable (*i.e.* the increment quotient).**

2°. Find the limit of this quotient when the increment of the variable approaches the limit zero.*

It must be emphasized here that the characteristic thing in differentiation is finding the limit of a quotient. From the standpoint of the Differential Calculus a function is of no interest if the limit mentioned does not exist. Functions possessing derivatives are said to be *differentiable*, and it is of prime importance to show, for example, that the elementary functions of § 2 are differentiable.

* The student must notice that the limit of the increment quotient cannot be found by Theorem III, § 8, since the limit of the denominator is zero.

15. Differentiation of the Elementary Functions x^m, $\sin x$, $\log_a x$.

(*a*) To prove $\dfrac{d}{dx}x^m = mx^{m-1}$.

Now $\Delta(x^m) = (x+h)^m - x^m$ if $\Delta x = h$.

But $(x+h)^m = x^m + mx^{m-1}h + \cdots + h^m$,

the terms not written containing powers of h.

$$\therefore\ \Delta(x^m) = mx^{m-1}h + \cdots + h^m;$$

$$\frac{\Delta(x^m)}{\Delta x} = mx^{m-1} + \cdots + h^{m-1},$$

where again the terms not written contain powers of h.

Putting $h = 0$, we find

$$(1)\qquad \frac{d}{dx}(x^m) = mx^{m-1}.$$

(*b*) To prove $\dfrac{d}{dx}\sin x = \cos x$.

Since $\Delta \sin x = \sin(x+h) - \sin x$

$$= 2\cos(x + \tfrac{1}{2}h)\sin\tfrac{1}{2}h \quad (\S\ 12,\ \text{Ex. } 5),$$

we find

$$\frac{\Delta \sin x}{\Delta x} = \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$= \cos(x+\tfrac{1}{2}h)\cdot\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}.$$

But $$\text{Limit}\left(\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right)_{h=0} = 1 \quad (\S\ 9),$$

and $$\text{Limit}\,(\cos(x+\tfrac{1}{2}h))_{h=0} = \cos x$$

(since $\cos x$ is continuous, § 6),

so that we may apply the theorem II, § 8, and we have

$$(2)\qquad \frac{d}{dx}\sin x = \cos x.$$

(*c*) To prove $\frac{d}{dx}\log_a x = \log_a e \frac{1}{x}$.

From § 12, Ex. 4, we have

$$\frac{\Delta \log_a x}{\Delta x} = \frac{\log_a\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}\log_a\left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right],$$

since the introduction of the exponent $\frac{x}{h}$ is, by the principles of logarithms, equivalent to multiplying the logarithm itself by that exponent.

Now $\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$ is the expression of § 11 if we write in that expression $z = \frac{x}{h}$.

Also $$\text{Limit}\left[\frac{x}{h}\right]_{h=0} = \infty,$$

hence $$\text{Limit}\left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]_{h=0} = \text{Limit}\left[\left(1+\frac{1}{z}\right)^{z}\right]_{z=\infty} = e.$$

$$\therefore \text{ Limit}\left[\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]_{h=0} = \log_a e,$$

(since $\log x$ is a continuous function),

and we have

(3) $$\frac{d}{dx}\log_a x = \log_a e \frac{1}{x}.$$

Formula (3) becomes most simple when $a = e$, for then

$$\frac{d}{dx}\log_e x = \frac{1}{x}.$$

Logarithms to the base e are called *natural logarithms* or *Napierian logarithms* (§ 10), and the factor $\log_a e$ in (3) is called the *modulus* of the system whose base is a, i.e. *the number by which natural logarithms must be multiplied in order to obtain logarithms to any given base* a.

We write $$M = \text{modulus} = \log_a e.$$

For example, the modulus of the *common* system of base 10 is $\log_{10} e$, and

$$\log_{10} e = 0.43429$$

to five decimal places.

If in (1), (2), and (3) we write u for x, we have

$$(4) \qquad \frac{d}{du} u^m = m u^{m-1}; \quad \frac{d}{du} \sin u = \cos u; \quad \frac{d}{du} \log_a u = \log_a e \frac{1}{u}.$$

EXERCISE 4

1. Differentiate with respect to x.

(*a*) $x^2 + 3ax + b$. *Ans.* $2x + 3a$.

(*b*) $\dfrac{a}{x+b}$. *Ans.* $-\dfrac{a}{(x+b)^2}$.

(*c*) $\sqrt{x}$ (cf. Ex. 4, p. 22). *Ans.* $\dfrac{1}{2\sqrt{x}}$.

2. Prove $\dfrac{d}{dx} \cos x = -\sin x$.

3. Prove $\dfrac{d}{dx}\sqrt{1+x} = \dfrac{1}{2\sqrt{1+x}}$.

4. Prove $\dfrac{d}{dv}\left(\dfrac{a}{v}\right) = -\dfrac{a}{v^2}$.

5. Prove $\dfrac{d}{du}(Cu) = C$, if C is any constant.

6. From the law of falling bodies

$$s = 16.1\, t^2,$$

we found (§ 14)

$$\frac{ds}{dt} = 32.2\, t,$$

or

$$\text{velocity} = v = 32.2\, t.$$

Prove

$$\frac{dv}{dt} = 32.2.$$

What does

$$\frac{dv}{dt} = \text{Limit}\left(\frac{\Delta v}{\Delta t}\right)_{\Delta t = 0}$$

represent in Mechanics? Acceleration. *Ans.*

7. Find the velocity and acceleration of the motion defined by

(1) $s = at + \frac{1}{2}gt^2$; *Ans.* $V = a + gt$; accel. $= g$.

(2) $s = at - \frac{1}{2}gt^2$; *Ans.* $V = a - gt$; accel. $= -g$.

8. Find the slope of the tangent to $y = 6x - x^2$ at the origin. *Ans.* 6.

16. Certain General Rules. We prove in this section several important rules for differentiation of a general character.

Let the variables u, v, w, etc., be functions of the variable x.

I. *To differentiate any algebraic sum of these variables.*

For example, to find $\frac{d}{dx}(u + v - w)$.

Now

$$\Delta(u+v-w) = u + \Delta u + v + \Delta v - (w + \Delta w) - (u + v - w),$$

$$= \Delta u + \Delta v - \Delta w,$$

$$\therefore \frac{\Delta(u + v - w)}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}.$$

Since $\text{Limit}\left[\frac{\Delta u}{\Delta x}\right]_{\Delta x=0} = \frac{du}{dx}$, $\text{Limit}\left[\frac{\Delta v}{\Delta x}\right]_{\Delta x=0} = \frac{dv}{dx}$,

$$\text{Limit}\left[\frac{\Delta w}{\Delta x}\right]_{\Delta x=0} = \frac{dw}{dx},$$

we may apply I, § 8, and we have

(5) $$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

II. *To differentiate a product.*

For example, to get $\frac{d}{dx}(uv)$.

Now $$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv,$$
$$= u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$
$$\therefore\ \frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Since $$\text{Limit}\left[\frac{\Delta v}{\Delta x}\right]_{\Delta x=0} = \frac{dv}{dx},\quad \text{Limit}\left[\frac{\Delta u}{\Delta x}\right]_{\Delta x=0} = \frac{du}{dx},$$
$$\text{Limit}\,[\Delta u]_{\Delta x=0} = 0,$$

we may apply I and II, § 8, and obtain

(6) $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

To find $\frac{d}{dx}(uvw)$, consider uvw as made up of the two factors uv and w; then, by (6),

$$\frac{d}{dx}(uv\cdot w) = uv\frac{dw}{dx} + w\frac{d(uv)}{dx},$$

or by (6) again,

(7) $$= uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}.$$

III. *To differentiate a quotient.*

Since $$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v},$$

$$\therefore\ \frac{\Delta}{\Delta x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v}.$$

Then since $\text{Limit}\,[v^2 + v\Delta v]_{\Delta x=0} = v^2$, we may apply I, II, III, § 8, and have

(8) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\dfrac{dv}{dx} - v\dfrac{du}{dx}}{v^2}.$$

From (6), (7), and (8) we have the rules:

I. The derivative of an algebraic sum of any number of variables is equal to the same algebraic sum of the derivatives of the variables.

II. The derivative of a product of any number of variables is equal to the sum of all the products formed by multiplying the derivative of each variable by all the remaining variables.

III. The derivative of a quotient equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

To these we may add the following:

IV. The derivative of a constant is zero.

V. The derivative of a constant times a variable equals the constant times the derivative of the variable.

Rule V comes from (6) and IV, if we place u equal to a constant.

EXAMPLES

1. Work out $\qquad \dfrac{d}{dx}(1+x^2)(1-2x^2).$

Rule II is first applied, and we get

$$\frac{d}{dx}(1+x^2)(1-2x^2)=(1+x^2)\frac{d}{dx}(1-2x^2)+(1-2x^2)\frac{d}{dx}(1+x^2).$$

By Rule I, $\qquad \dfrac{d}{dx}(1-2x^2)=\dfrac{d}{dx}(1)-\dfrac{d}{dx}(2x^2),$

$$\frac{d}{dx}(1+x^2) = \frac{d}{dx}(1)+\frac{d}{dx}(x^2).$$

Since by V, $\dfrac{d}{dx}(2x^2)=2\dfrac{d}{dx}x^2$, and from (4) § 15, $\dfrac{d}{dx}x^2=2x$,

we have finally,

$$\frac{d}{dx}(1+x^2)(1-2x^2)=(1+x^2)\cdot -4x+(1-2x^2)\cdot 2x=-2x(1+4x^2).$$

2. Work out $$\frac{d}{dx}\left(\frac{\sin x}{\log_e x}\right).$$

Rule III we use first and find

$$\frac{d}{dx}\left(\frac{\sin x}{\log_e x}\right)=\frac{\log_e x\frac{d}{dx}\sin x-\sin x\frac{d}{dx}\log_e x}{(\log_e x)^2}.$$

By (4) § 15, $\quad \dfrac{d}{dx}\sin x=\cos x,\ \dfrac{d}{dx}\log_e x=\dfrac{1}{x}.$

$$\therefore\ \frac{d}{dx}\left(\frac{\sin x}{\log_e x}\right)=\frac{x\cos x\log_e x-\sin x}{x(\log_e x)^2}.$$

EXERCISE 5

Prove the following differentiations :

1. $\dfrac{d}{dx}x(1-x)=1-2x.$

3. $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)=\dfrac{x\cos x-\sin x}{x^2}.$

2. $\dfrac{d}{dx}\left(\dfrac{x}{1+x^2}\right)=\dfrac{1-x^2}{(1+x^2)^2}.$

4. $\dfrac{d}{dx}\left(\dfrac{1+x^2}{1-x^2}\right)=\dfrac{4x}{(1-x^2)^2}.$

5. $\dfrac{d}{dx}\left(\dfrac{x^m}{1+x^n}\right)=\dfrac{x^{m-1}(m+x^n(m-n))}{(1+x^n)^2}.$

6. $\dfrac{d}{dx}(x^m\log x)=x^{m-1}(1+m\log x).$

7. $\dfrac{d}{dx}\left(\dfrac{2x}{1-x^2}\right)=\dfrac{2(1+x^2)}{(1-x^2)^2}.$

8. $\dfrac{d}{dx}\left(\dfrac{x^n}{a}\right)=\dfrac{n}{a}x^{n-1}.$

9. $\dfrac{d}{dx}\left(\dfrac{a}{x^n}\right)=\dfrac{-na}{x^{n+1}}.$

10. $\dfrac{d}{dx}x^m(1-x)^n=x^{m-1}(1-x)^{n-1}(m-(m+n)x).$

Special attention should be given to the following:

11. Find $\quad \dfrac{d}{du}\tan u.\quad$ Since $\tan u=\dfrac{\sin u}{\cos u},$

we have $$\frac{d}{du}\tan u=\frac{d}{du}\left(\frac{\sin u}{\cos u}\right).$$

Applying III (4), § 15, and Example 2, § 15, we find

$$\frac{d}{du}\tan u=\sec^2 u.$$

Prove

12. $\frac{d}{du}\cot u = -\operatorname{cosec}^2 u.$

13. $\frac{d}{du}\sec u = \sec u \tan u.$ $\left(\text{Put } \sec u = \frac{1}{\cos u}.\right)$

14. $\frac{d}{du}\csc u = -\csc u \cot u.$

17. We come now to two most important rules.

Differentiation of Inverse Functions. Suppose y is a function of x, *i.e.* in symbols

(9) $$y = f(x).$$

Then it is usually possible inversely to calculate x when values are assumed for y, *i.e.* we may choose y for the independent variable instead of x, so that by solving (1) for x we obtain

(10) $$x = \phi(y).$$

Then $f(x)$ and $\phi(y)$ are called *inverse functions.*

Example. If $y = a^x$, then $x = \log_a y$; that is, a^x and $\log_a y$ are inverse functions.

Let now Δx and Δy be corresponding increments of x and y, so that Δx and Δy vanish together, since we are dealing here with continuous functions. Then the increment quotient is $\frac{\Delta y}{\Delta x}$ or $\frac{\Delta x}{\Delta y}$, according as x or y is taken for the independent variable.

Now by multiplication, $\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y} = 1,$

hence $$\text{Limit}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x=0}\cdot \text{Limit}\left(\frac{\Delta x}{\Delta y}\right)_{\Delta y=0} = 1,$$

by II, § 8, since, as above emphasized, Δy and Δx vanish together. We have, therefore, in the derivative notation,

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1, \text{ or solving,}$$

(11) $$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.*$$

VI. If y is a function of x, and inversely x a function of y, then the derivative of x with respect to y equals the reciprocal of the derivative of y with respect to x.

Differentiation of a Function of a Function. We have seen by (4) how to differentiate with respect to x the elementary function $\sin x$. Suppose we wish to find

$$\frac{d}{dx}\sin(1+x^2),$$

for which the rule (4) does not suffice. We then introduce the variable $u=1+x^2$, and setting $y=\sin(1+x^2)=\sin u$, we have before us the relations

(12) $$y=\sin u,\quad u=1+x^2,$$

and we say *y is a function of x through u*, i.e. y is a *function of a function.*

Now, if Δy, Δu, and Δx are corresponding increments of y, u, and x, then forming the increment quotients $\frac{\Delta y}{\Delta u}$, $\frac{\Delta u}{\Delta x}$, we have, by multiplication,

(13) $$\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}=\frac{\Delta y}{\Delta x}.$$

* The student will not fail to notice that in (11) the familiar property of a fraction, $\frac{a}{b}=1\div\frac{b}{a}$ is suggested. But he must not forget that $\frac{dy}{dx}$ is *not* a fraction, but merely the symbol for the limiting value of the fraction $\frac{\Delta y}{\Delta x}$.

But the increments Δy, Δu, and Δx vanish together, so that, by II, § 8,

$$\text{Limit}\left(\frac{\Delta y}{\Delta u}\right)_{\Delta u=0} \cdot \text{Limit}\left(\frac{\Delta u}{\Delta x}\right)_{\Delta x=0} = \text{Limit}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x=0},$$

or (14)

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

VII. If y is a function of x through u, then the derivative of y with respect to x equals the products of the derivatives of y with respect to u and of u with respect to x.

Thus in (12), since $\frac{d}{du}\sin u = \cos u$, $\frac{d}{dx}(1 + x^2) = 2x$, we find $\frac{d}{dx}\sin(1 + x^2) = \cos u \cdot 2x = 2x\cos(1 + x^2)$.

EXERCISE 6

1. Show that the geometrical significance of (11) is that the tangent makes complementary angles with XX' and YY'.

2. If a material point P, whose rectangular coördinates are x and y, move in a plane, then x and y are functions of the time t. Now the

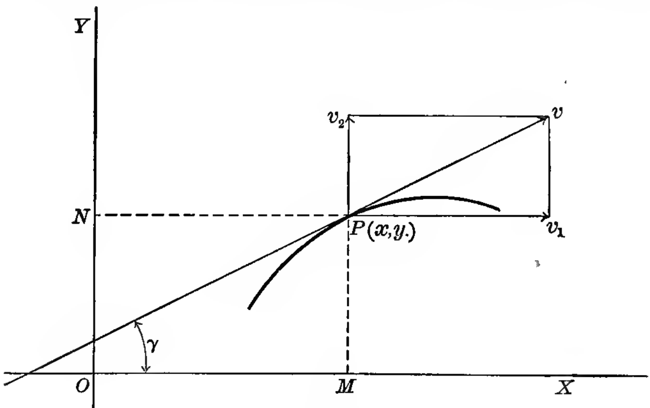


FIG. 11

horizontal component v_1 (see Fig. 11) of the velocity v is the velocity along OX of the projection M of P, and is therefore the time rate of

change of x, or $v_1 = \frac{dx}{dt}$. In the same manner, the vertical component v_2 equals $\frac{dy}{dt}$; and since

$$v = \sqrt{v_1^2 + v_2^2},$$

we have
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

For the *direction* of the velocity, $\tan\gamma = \frac{v_2}{v_1}$, or

$$\tan\gamma = \frac{dy}{dt} \div \frac{dx}{dt}.$$

3. Prove that the equations $x = a\cos t$, $y = a\sin t$ define uniform motion in the circle $x^2 + y^2 = a^2$.

4. If the coördinates (x, y) of a point on a curve are functions of a variable θ, show that

$$(15) \qquad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}.$$

(Use (14) and then (11).)

5. To find $\frac{d}{dx}(x^{\frac{1}{q}})$ when q is any positive integer; *i.e.* to *differentiate any root of* x.

Put $u = x^{\frac{1}{q}}$, then $x = u^q$; hence, by (4), § 15,

$$\frac{dx}{du} = qu^{q-1},$$

and by (11)
$$\frac{du}{dx} = \frac{1}{qu^{q-1}}.$$

But
$$u^{q-1} = x^{\frac{q-1}{q}} = x^{1-\frac{1}{q}}.$$

$$\therefore \frac{du}{dx} = \frac{1}{q}x^{\frac{1}{q}-1}, \text{ or } \frac{d}{dx}(x^{\frac{1}{q}}) = \frac{1}{q}x^{\frac{1}{q}-1};$$

i.e. *the same rule holds for roots and powers of the independent variable.*

6. From (4), § 15, and VII, we have

$$(16)\quad \begin{cases} \dfrac{d}{dx}(u^m) = \dfrac{d}{du}(u^m)\dfrac{du}{dx} = mu^{m-1}\dfrac{du}{dx}. \\[2ex] \dfrac{d}{dx}\sin u = \dfrac{d}{du}(\sin u)\dfrac{du}{dx} = \cos u\dfrac{du}{dx}. \\[2ex] \dfrac{d}{dx}\log_a u = \dfrac{d}{du}(\log_a u)\dfrac{du}{dx} = \log_a e\dfrac{\frac{du}{dx}}{u}. \end{cases}$$

7. To find $\dfrac{d}{dx}(x^{\frac{p}{q}})$.

Letting $u = x^{\frac{1}{q}}$ (Example 3), we have

$$u^p = x^{\frac{p}{q}}.$$

$$\therefore \frac{d}{dx}(x^{\frac{p}{q}}) = \frac{d}{dx}u^p = pu^{p-1}\frac{du}{dx} \text{ (Example 6),}$$

$$= px^{\frac{p-1}{q}} \cdot \frac{d}{dx}(x^{\frac{1}{q}}),$$

$$= px^{\frac{p-1}{q}} \cdot \frac{1}{q}x^{\frac{1}{q}-1} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

Hence the rule of (4), § 15, *for powers holds for any commensurable exponent.*

8. To prove $\dfrac{d}{dx}\operatorname{arc}\sin u = \dfrac{\frac{du}{dx}}{\sqrt{1-u^2}}$.

Placing $y = \operatorname{arc}\sin u$, we have inversely,

$$u = \sin y.$$

$$\therefore \frac{du}{dy} = \cos y, \text{ and by (15), } \frac{dy}{du} = \frac{1}{\cos y}.$$

But $$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-u^2}.$$

Hence $$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}},$$

and by (16), $$\frac{dy}{dx} = \frac{d}{dx}\operatorname{arc}\sin u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

9. Prove $\frac{d}{dx}\operatorname{arc}\tan u = \frac{\frac{du}{dx}}{1+u^2}$.

(Remember $\sec^2 y = 1 + \tan^2 y$.)

10. Prove $\frac{d}{dx}\operatorname{arc}\sec u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}$.

11. Prove $\frac{d}{dx}a^u = a^u \log_e a \frac{du}{dx}$.

Putting $y = a^u$, we have inversely,

$$u = \log_a y.$$

$$\therefore \frac{du}{dy} = \log_a e \frac{1}{y} \quad (\S\ 15, (4)).$$

$$\therefore \frac{dy}{du} = \frac{y}{\log_a e} = \frac{a^u}{\log_a e}.$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}a^u = a^u \log_e a \frac{du}{dx}.^*$$

In particular,
$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

Example 11 shows that the exponential function e^x possesses the remarkable property of being *its own derivative*, for

$$\frac{d}{dx}e^x = e^x \frac{dx}{dx} = e^x.$$

In general, if a is any constant, then

(1)
$$\frac{d}{dx}e^{ax} = ae^{ax}, \text{ since } \frac{d}{dx}(ax) = a;$$

that is, the derivative of the function e^{ax} is *proportional* to (*i.e.* a times) the function itself. For a reason now to be explained, the function e^{ax} is said to follow the *Compound Interest Law.*

If P dollars be drawing compound interest at r per cent, then in the time Δt the interest is $\frac{r}{100}P\Delta t$, and hence the change in P or ΔP is given by

$$\Delta P = \frac{r}{100}P\Delta t, \text{ or } \frac{\Delta P}{\Delta t} = \frac{r}{100}P.$$

* From the principle in logarithms, $\log_e a = \frac{1}{\log_a e}$.

Now suppose the interest to be added on *continuously*, and not after *finite* intervals of time Δt, *i.e.* we make Δt approach the limit zero, and conceive of P as increasing *continuously*; then

$$\frac{dP}{dt} = \frac{r}{100} P,$$

so that a sum of money accumulating continuously at compound interest has precisely the property above enunciated in (1), viz. *its derivative is proportional to the sum itself.*

18. From the examples in Exercises 5 and 6 and the Rule VII, we deduce the following fundamental formulæ for differentiation:

VIII. $\dfrac{d}{dx} u^m = m u^{m-1} \dfrac{du}{dx}$ (**m any commensurable number**).

IX. $\dfrac{d}{dx} \log_a u = \log_a e \dfrac{\frac{du}{dx}}{u}$.

X. $\dfrac{d}{dx} a^u = a^u \log_e a \dfrac{du}{dx}$ (**a any positive constant**).

XI. $\dfrac{d}{dx} \sin u = \cos u \dfrac{du}{dx}$.

XII. $\dfrac{d}{dx} \cos u = - \sin u \dfrac{du}{dx}$.

XIII. $\dfrac{d}{dx} \tan u = \sec^2 u \dfrac{du}{dx}$.

XIV. $\dfrac{d}{dx} \cot u = - \csc^2 u \dfrac{du}{dx}$.

XV. $\dfrac{d}{dx} \sec u = \sec u \tan u \dfrac{du}{dx}$.

XVI. $\dfrac{d}{dx} \csc u = - \csc u \cot u \dfrac{du}{dx}$.

XVII. $\dfrac{d}{dx} \operatorname{arc} \sin u = \dfrac{\frac{du}{dx}}{\sqrt{1-u^2}}$.

XVIII. $\dfrac{d}{dx} \operatorname{arc} \cos u = - \dfrac{\frac{du}{dx}}{\sqrt{1-u^2}}$.

$$\textbf{XIX.}\quad \frac{d}{dx}\operatorname{arc}\tan u = \frac{\dfrac{du}{dx}}{1+u^2}.$$

$$\textbf{XX.}\quad \frac{d}{dx}\operatorname{arc}\cot u = -\frac{\dfrac{du}{dx}}{1+u^2}.$$

$$\textbf{XXI.}\quad \frac{d}{dx}\operatorname{arc}\sec u = \frac{\dfrac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\textbf{XXII.}\quad \frac{d}{dx}\operatorname{arc}\csc u = -\frac{\dfrac{du}{dx}}{u\sqrt{u^2-1}}.$$

The formulæ and rules I–XXII the student must memorize. With their aid differentiation of the commoner functions is made rapid and easy, but perfect familiarity with them is indispensable.

To show the application of the rules three examples are now given:

1. Find $\dfrac{d}{dx}\left(\dfrac{1-x}{\sqrt{1+x^2}}\right)$.

By III, $$\frac{d}{dx}\left(\frac{1-x}{\sqrt{1+x^2}}\right) = \frac{\sqrt{1+x^2}\dfrac{d}{dx}(1-x) - (1-x)\dfrac{d}{dx}\sqrt{1+x^2}}{1+x^2}.$$

By I and IV, $\dfrac{d}{dx}(1-x) = -1$;

from VIII, $$\frac{d}{dx}(1+x^2)^{\frac{1}{2}} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\frac{d}{dx}(1+x^2),$$

and since $\dfrac{d}{dx}(1+x^2) = 2x$, we have

$$\frac{d}{dx}\left(\frac{1-x}{\sqrt{1+x^2}}\right) = \frac{\sqrt{1+x^2}\cdot -1 - (1-x)(1+x^2)^{-\frac{1}{2}}\cdot x}{(1+x^2)}.$$

To simplify, multiply numerator and denominator by

$$(1+x^2)^{\frac{1}{2}}.$$

Then, since $(1 + x^2)^0 = 1$, we have, reducing,

$$\frac{d}{dx}\left(\frac{1 - x}{\sqrt{1 + x}}\right) = -\frac{1 + x}{(1 + x^2)^{\frac{3}{2}}}.$$

2. Find $\frac{d}{dx}\log_e\sqrt{\frac{1 - \cos x}{1 + \cos x}}$.

For convenience, set $y = \log_e\sqrt{\frac{1 - \cos x}{1 + \cos x}}$.

Since $\log_e\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1}{2}\log(1 - \cos x) - \frac{1}{2}\log(1 + \cos x)$,
then by I and V,

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\log_e(1 - \cos x) - \frac{1}{2}\frac{d}{dx}\log_e(1 + \cos x).$$

Applying IX, we have

$$\frac{dy}{dx} = \frac{1}{2}\frac{\frac{d}{dx}(1 - \cos x)}{1 - \cos x} - \frac{1}{2}\frac{\frac{d}{dx}(1 + \cos x)}{1 + \cos x}, \text{ and by XII,}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\sin x}{1 - \cos x} + \frac{\sin x}{1 + \cos x}\right) = \frac{\sin x}{1 - \cos^2 x} = \frac{1}{\sin x}.$$

$$\therefore \frac{d}{dx}\log_e\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1}{\sin x}.$$

3. Find $\frac{d}{d\theta}\arctan\left(\frac{e^\theta - e^{-\theta}}{2}\right)$.

Setting the function equal to y, we have, by XIX,

$$\frac{dy}{d\theta} = \frac{\frac{d}{d\theta}\left(\frac{e^\theta - e^{-\theta}}{2}\right)}{1 + \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2} = \frac{\frac{e^\theta + e^{-\theta}}{2}}{\frac{4 + e^{2\theta} - 2 + e^{-2\theta}}{4}} \text{ (by X)}$$

$$= \frac{2}{e^\theta + e^{-\theta}}.$$

EXERCISE 7

Prove the following differentiations:

1. $\dfrac{d}{dx}(x^2+1)\sqrt{x^3-x}=\dfrac{7x^4-2x^2-1}{2(x^3-x)^{\frac{1}{2}}}.$

2. $\dfrac{d}{dx}\sqrt{\dfrac{1+x}{1-x}}=\dfrac{1}{(1-x)\sqrt{1-x^2}}.$

3. $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{1-x^2}}\right)=\dfrac{1}{(1-x^2)^{\frac{3}{2}}}.$

4. $\dfrac{d}{dx}\left(\dfrac{3x^3+2}{x(x^3+1)^{\frac{2}{3}}}\right)=-\dfrac{2}{x^2(x^3+1)^{\frac{5}{3}}}.$

5. $\dfrac{d}{dx}(1-2x+3x^2-4x^3)(1+x)^2=-20x^3(1+x).$

6. $\dfrac{d}{dx}(1-3x^2+6x^4)(1+x^2)^3=60x^5(1+x^2)^2.$

7. $\dfrac{d}{dx}(5^{x^2+2x})=2(x+1)5^{x^2+2x}\log_e 5.$

8. $\dfrac{d}{dx}x^n a^x=x^{n-1}a^x(n+x\log_e a).$

9. $\dfrac{d}{dx}\left[\dfrac{x\log_e x}{1-x}+\log_e(1-x)\right]=\dfrac{\log_e x}{(1-x)^2}.$

10. $\dfrac{d}{dx}e^{ax}\left(x^3-\dfrac{3x^2}{a}+\dfrac{6x}{a^2}-\dfrac{6}{a^3}\right)=ax^3e^{ax}.$

11. $\dfrac{d}{dx}\log_e(e^x+e^{-x})=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}.$

12. $\dfrac{d}{dx}(\sqrt{x}-\log_e(1+\sqrt{x}))=\dfrac{1}{2(1+\sqrt{x})}.$

13. $\dfrac{d}{d\theta}\tan^2 5\theta=10\tan 5\theta\sec^2 5\theta.$

14. $\dfrac{d}{d\theta}\sin^3\theta\cos\theta=\sin^2\theta(3\cos^2\theta-\sin^2\theta).$

15. $\dfrac{d}{d\theta}\log\sec\theta=\tan\theta.$

16. $\frac{d}{d\theta}(\tan^2\theta - \log\sec^2\theta) = 2\tan^3\theta.$

17. $\frac{d}{d\theta}\sin n\theta\sin^n\theta = n\sin^{n-1}\theta\sin(n+1)\theta.$

18. $\frac{d}{dx}\arcsin(3x - 4x^3) = \frac{3}{\sqrt{1-x^2}}.$

19. $\frac{d}{dx}\operatorname{arc\,sec}\frac{x}{a} = \frac{a}{x\sqrt{x^2-a^2}}.$

20. $\frac{d}{dx}\operatorname{arc\,csc}\frac{1}{2x^2-1} = \frac{2}{\sqrt{1-x^2}}.$

21. $\frac{d}{dx}\arcsin\frac{1-x^2}{1+x^2} = -\frac{2}{1+x^2}.$

22. $\frac{d}{dx}\arctan\frac{x+a}{1-ax} = \frac{1}{1+x^2}.$

23. $\frac{d}{dx}\arccos\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{-2}{e^x+e^{-x}}.$

24. $\frac{d}{dx}\operatorname{arc\,sec}\sqrt{\frac{2}{1+x}} = \frac{-1}{2\sqrt{1-x^2}}.$

25. $\frac{d}{dx}\left(\operatorname{arc\,cot}\frac{a}{x} + \log_e\sqrt{\frac{x-a}{x+a}}\right) = \frac{2ax^2}{x^4-a^4}.$

19. Differentiation of Implicit Functions. If an analytic relation is given between two variables not solved for either variable in terms of the other, then either variable is said to be an *implicit function* of the other.

For example, in $x^2 - y^2 + 9 = 0$ either x or y is an implicit function of the other variable.

In such a case either variable may be chosen for the *independent* variable, and if we can solve explicitly for the other (as in the above example for y, giving $y = \sqrt{x^2-9}$), then we can differentiate as before. But it is generally better not to solve the equation, but to differentiate the given relation as it stands.

Thus, to find $\frac{dy}{dx}$ from

$$x^2 - 3xy + 2y^2 = 3.$$

Then
$$\frac{d}{dx}(x^2) - 3\frac{d}{dx}(xy) + 2\frac{d}{dx}(y^2) = \frac{d}{dx}3.$$

$$\therefore 2x - 3\left(y + x\frac{dy}{dx}\right) + 4y\frac{dy}{dx} = 0,$$

and
$$\frac{dy}{dx} = \frac{2x - 3y}{3x - 4y}.$$

To justify this process is beyond the limits of this text-book. One thing is to be noted, namely, that only those values of the variables which satisfy the original relation can be substituted for the derivative.

EXERCISE 8

Find $\frac{dy}{dx}$ from the following equations:

1. $y^2 - 2xy = a^2$. *Ans.* $\frac{dy}{dx} = \frac{y}{y - x}$.

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

3. $ax^2 + 2bxy + cy^2 + 2fx + 2gy + h = 0$.
Ans. $\frac{dy}{dx} = -\frac{ax + by + f}{bx + cy + g}$.

4. $x^3 + y^3 - 3axy = 0$. *Ans.* $\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}$.

5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

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6. Given $r = a(1 - \cos\theta)$; show $\frac{dr}{d\theta} = a\sin\theta$.

7. Given $r^2 = a^2\cos 2\theta$; show $\frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$.

20. Derivatives of Higher Orders. Since the derivative of a function of a variable x with respect to x is also in general a function of x, we may differentiate the derivative itself, that is, carry out the operation,

$$\frac{d}{dx}\left(\frac{d}{dx}f(x)\right).$$

This double operation is indicated by the more compact notation,

$$\frac{d^2}{dx^2}f(x),$$

and this new function is called the *second derivative*. In the same way,

$$\frac{d}{dx}\frac{d^2}{dx^2}f(x) \equiv \frac{d^3}{dx^3}f(x)$$

is the *third derivative*, and in general,

$$\frac{d^n}{dx^n}f(x)$$

is the nth derivative of $f(x)$, that is, the result of differentiating $f(x)$ n times. The following notation is also used,

$$\frac{d}{dx}f(x) = f'(x),\quad \frac{d^2}{dx^2}f(x) = f''(x),\quad \cdots,\quad \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$

The operation of finding the successive derivatives is called *successive differentiation.*

EXERCISE 9

1. Given $\qquad f(x) = 3x^4 - 4x^2 + 6x - 1,$

then $\qquad f'(x) = 12x^3 - 8x + 6;$

$\qquad f''(x) = 36x^2 - 8,$ etc.

2. Given $\quad f(x)=e^{ax}$; prove $f^{(n)}(x)=a^n e^{ax}$.

3. Given $\quad f(x)=\log_e(1-x)$; prove $f^{(n)}(x)=\dfrac{(-1)^n \lfloor n-1}{(1-x)^n}$.

4. Given $\quad y=x^3\log_e x$; prove $\dfrac{d^4y}{dx^4}=\dfrac{6}{x}$.

5. Given $\quad y=\log_e \sin x$; find $\dfrac{d^3y}{dx^3}=\dfrac{2\cos x}{\sin^3 x}$.

6. Given $\quad y=e^{2x}(x^2-3x+3)$; find $\dfrac{d^3y}{dx^3}=8x^2e^{2x}$.

7. Given $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, or $b^2x^2+a^2y^2=a^2b^2$, to find $\dfrac{d^2y}{dx^2}$.

From Ex. 2, Exercise 8,

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}; \quad \therefore \quad \frac{d^2y}{dx^2}=-\frac{a^2y\dfrac{d}{dx}(b^2x)-b^2x\dfrac{d}{dx}(a^2y)}{a^4y^2},$$

or

$$\frac{d^2y}{dx^2}=-\frac{a^2b^2y-b^2a^2x\dfrac{dy}{dx}}{a^4y^2};$$

then substituting for $\dfrac{dy}{dx}$ and reducing,

$$\frac{d^2y}{dx^2}=-\frac{b^2(a^2y^2+b^2x^2)}{a^4y^3}=-\frac{b^4}{a^2y^3}.$$

8. From $\quad y^2=4px$, find $\dfrac{d^2y}{dx^2}=-\dfrac{4p^2}{y^3}=-\dfrac{p}{xy}$.

9. From $\quad y^2-2xy=a^2$, prove $\dfrac{d^2y}{dx^2}=\dfrac{a^2}{(y-x)^3}$.

CHAPTER III

APPLICATIONS

21. Tangent and Normal. For all applications of the Calculus to Geometry the fact established in § 14 is of fundamental importance, viz.

Theorem. *The value of the derivative of y with respect to x found from the equation of a curve in rectangular coördinates gives the slope of the tangent at any point on that curve*, or

$$\frac{dy}{dx} = \text{slope of tangent.}$$

If we wish the slope at any particular point (x', y'), we have to substitute x' and y' respectively for x and y in the general expression for $\frac{dy}{dx}$. Let $\left(\frac{dy}{dx}\right)'$ be the value of $\frac{dy}{dx}$ after this substitution, then we have from Analytic Geometry,

Equation of the tangent at (x', y') is

$$(17) \qquad y - y' = \left(\frac{dy}{dx}\right)'(x - x').$$

Since the normal is perpendicular to the tangent, and from (11), $\left(\frac{dy}{dx}\right)' = \frac{1}{\left(\frac{dx}{dy}\right)'}$, we find

Equation of the normal at (x', y') is

$$(18) \qquad y - y' = -\left(\frac{dx}{dy}\right)'(x - x').$$

EXERCISE 10

1. Find equations of tangent and normal to the parabola $y^2=4x+1$ at the point whose ordinate is 3.

Substituting 3 for y, we find $x=2$, hence (x', y') is $(2, 3)$. Differentiating, $\dfrac{dy}{dx}=\dfrac{2}{y}$. $\therefore \left(\dfrac{dy}{dx}\right)'=\dfrac{2}{3}$.

Ans. tangent, $2x-3y+5=0$; normal, $3x+2y-12=0$.

2. Find equation of tangent to the ellipse $b^2x^2+a^2y^2=a^2b^2$ at (x', y'). *Ans.* $b^2x'x+a^2y'y=a^2b^2$.

3. Show (Fig. 12) that the *subtangent* $M_1T_1=-y'\left(\dfrac{dx}{dy}\right)'$, and the *subnormal* $M_1N_1=y'\left(\dfrac{dy}{dx}\right)'$.

4. Prove that the subnormal in the parabola $y^2=4px$ has the constant length $2p$.

22. Sign of the Derivative. An important question is the following:

Is the function increasing or decreasing as the variable passes through a given value a?

The phrase "passing through a" is understood to mean that the series of values assumed by the variable is an

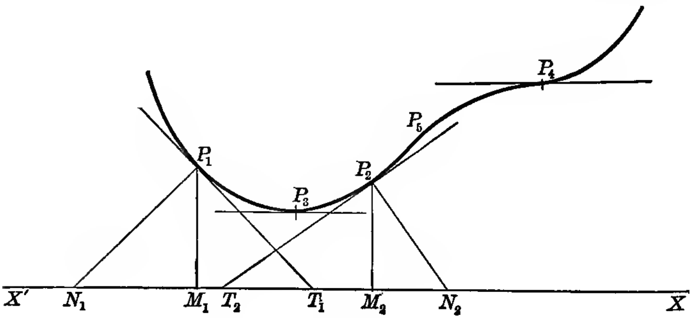

FIG. 12

increasing sequence including a, *i.e.* on the graph of the function we proceed from *left to right.* In Fig. 12, as we

pass through P_1 the ordinates are decreasing, while at P_2 the ordinates are increasing, and since the ordinates represent the values of the function and $\frac{dy}{dx}$ or $f'(x)$ is the slope of the tangent, we have the result:

The function $f(x)$ is increasing or decreasing as x passes through a according as $f'(a)$ is greater or less than zero.

At P_3 and P_4 (Fig. 12) the tangent is parallel to XX', and therefore $f'(x)$ vanishes at these points. For such values of x, therefore, the rule just given does not enable us to answer the question proposed.

If, now, for any value of x, say $x=a$, the second derivative $\frac{d^2y}{dx^2}$, or $f''(x)$, is positive, then as x passes through a, the first derivative $f'(x)$, or $\tan\gamma$, must be an increasing function of x, *i.e.* γ must be increasing as x passes through a; and therefore as we pass along the curve from left to right, the tangent is rotating *counter-clockwise*, and the curve is accordingly *concave upward* (as at (a), Fig. 13).

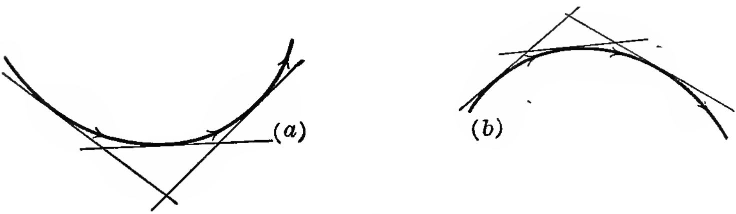

FIG. 13

On the contrary, if $f''(a) < 0$, the reasoning shows the tangent to be rotating *clockwise* as we pass along the graph through $x = a$, and hence the curve is *concave downward* ((*b*), Fig. 13).

The result is:

A curve is concave upward or downward as x passes through a according as the value of the second derivative $\frac{d^2y}{dx^2}$ for $x = a$ is greater or less than zero.

As before, if $\frac{d^2y}{dx^2} = 0$, the rule just given does not enable us to decide. If $\frac{d^2y}{dx^2} = 0$ for $x = a$ and *changes sign* as x passes through a, then at $x = a$ we have a point of inflection (P_4 and P_5, Fig. 12).

EXERCISE 11

1. Show that the following functions are either always increasing or always decreasing, and draw the graphs in each case:

$$(a)\ \tan x; \quad (b)\ e^x; \quad (c)\ \log x; \quad (d)\ \frac{1}{x}.$$

2. Show that $y = \sin x$ has a point of inflection at each intersection with XX'.

3. Determine the points of inflection of $y = (x - a)^3 + b$.

Ans. (a, b).

23. Maxima and Minima. A function $f(x)$ is said to be a *maximum* for $x = a$ when $f(a)$ is the *greatest* value of $f(x)$ as x passes through a.

A function $f(x)$ is said to be a *minimum* for $x = a$ when $f(a)$ is the *least* value of $f(x)$ as x passes through a.

In other words, a *maximum* value is greater than any other *in the immediate vicinity*, and similarly for a *minimum* value. It is not to be inferred that a maximum value is the greatest of *all* values of the function; on the contrary, a function may have several maxima.

Graphically, at a maximum we have a *highest* point (P_1 and P_3, Fig. 14), at a minimum a *lowest* point (P_2 and P_4).

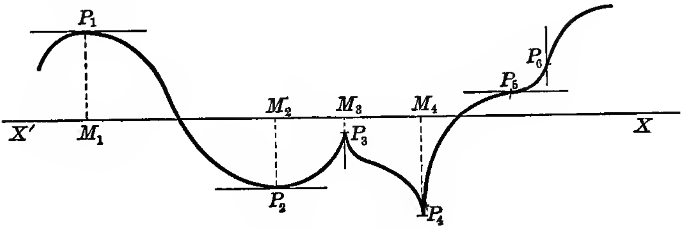

FIG. 14

Since, by definition, if $f(a)$ is a maximum, $f(x)$ must be an *increasing* function for $x < a$ and a decreasing function for $x > a$, we have (§ 22):

Theorem. *If $f(a)$ is a maximum value of $f(x)$, then the first derivative $f'(x)$ must change sign from positive to negative as x passes through a.*

By similar reasoning for a minimum, we find a change in sign from negative to positive must occur in $f'(x)$.

In either case, therefore, $f'(x)$ must change sign. If we now assume that $f'(x)$ is *continuous* for $x = a$, we see that $f'(a) = 0$; that is, the tangent at a highest or lowest point must be *horizontal* (P_1 and P_2 in Fig. 14). If, on the contrary, $f'(x)$ is *not* continuous for $x = a$, then the change in sign occurs by passage through ∞; *i.e.* the tangent becomes parallel to YY', as at P_3 and P_4. This case is, however, of minor importance, and is omitted from further consideration.

Furthermore, if $f''(a) < 0$, the curve at $x = a$ is concave downward, and we have a highest point (P_1), while $f''(a) > 0$ indicates a lowest point (P_2).

We have therefore the following

Rule for determination of Maximum and Minimum values of a function $f(x)$.

Find the first derivative $f'(x)$, and get the roots of the equation $f'(x) = 0$.

First Test. **If $f'(x)$ changes sign as x passes through any root a of the equation $f'(x) = 0$, then $f(a)$ is a maximum or minimum value according as the change is from $+$ to $-$, or from $-$ to $+$.**

Second Test. **Find the second derivative $f''(x)$; then, if a is any root of $f'(x) = 0$, $f(a)$ is a maximum if $f''(a) < 0$, and a minimum if $f''(a) > 0$. If, however, $f''(a) = 0$, we must use the first test.**

EXAMPLES

1. Examine the function $x^3 - 3x^2 - 9x + 5$ for maxima and minima.

Placing
$$f(x) \equiv x^3 - 3x^2 - 9x + 5,$$
then
$$f'(x) = 3x^2 - 6x - 9,$$
and the roots of $3x^2 - 6x - 9 = 0$ are $x = 3$ and -1.

Now $f''(x) \equiv 6x - 6$, and $f''(3) = 12$, $f''(-1) = -12$,

hence by the Rule, Second Test,

$$f(3) = -22 \text{ is a } \textit{minimum} \text{ value,}$$

and
$$f(-1) = 10 \text{ is a } \textit{maximum} \text{ value of the function.}$$

The student should draw the graph.

2. Examine the function $\dfrac{(x-1)^2}{(x+1)^3}$ for maxima and minima.

Here
$$f(x) \equiv \frac{(x-1)^2}{(x+1)^3}.$$

Differentiating and reducing, we find

$$f'(x) \equiv -\frac{(x-1)(x-5)}{(x+1)^4}.$$

The roots of $f'(x) = 0$ are therefore $x = 1$, $x = 5$. We now apply the First Test, since it is unwise to form the second derivative.

Taking account of the signs only, we have

$$\left.\begin{array}{l} \text{When* } x < 1,\ f'(x) = -\dfrac{(-)(-)}{+} = - \\[2ex] \text{When } x > 1,\ f'(x) = -\dfrac{(+)(-)}{+} = + \end{array}\right\} \begin{array}{l}\text{Hence } f(x) \text{ is a} \\ \textit{minimum} \text{ when } x = 1.\end{array}$$

$$\left.\begin{array}{l} \text{When } x < 5,\ f'(x) = -\dfrac{(+)(-)}{+} = + \\[2ex] \text{When } x > 5,\ f'(x) = -\dfrac{(+)(+)}{+} = - \end{array}\right\} \begin{array}{l}\text{Hence } f'(x) \text{ is a} \\ \textit{maximum} \text{ when } x = 5.\end{array}$$

$\therefore f(1) = 0$ is a minimum, and $f(5) = \frac{1}{8}$ a maximum value of the function.

EXERCISE 12

1. Examine the following functions for maxima and minima:

(*a*) $x^2 - 3x + 5$. Minimum value 11. *Ans.*

(*b*) $\dfrac{x}{1 + x^2}$. Max. value $\frac{1}{2}$, min. value $-\frac{1}{2}$. *Ans.*

(*c*) $6x + 3x^2 - 4x^3$. Max. value 5, min. value $-\frac{7}{4}$. *Ans.*

(*d*) $x^3 - 3x^2 + 6x$. No max. or min. values. *Ans.*

(*e*) $ax^2 + bx$. If $a > 0$, min. value $\dfrac{ac - b^2}{a}$, if $a < 0$, then $\dfrac{ac - b^2}{a}$ is a maximum. *Ans.*

(*f*) $10\sqrt{8x - x^2}$. Max. value 40. *Ans.*

This function is a maximum or minimum according as $8x - x^2$ is a maximum or minimum, hence† *a constant factor or a radical sign may be dropped in investigations of this sort.*

* We consider values of x differing only *very slightly* from the number on the right of the inequality sign.

† If u is any polynomial in x containing *no multiple factors*, we may show that $\sqrt{u}$ is a maximum or minimum only when u is a maximum or minimum. For if

2. Divide the number a into two such parts that their product shall be a maximum.

HINT. If x is one part, then $a-x$ is the other, and the function to be examined is $x(a-x)$ or $ax-x^2$. Equal parts. *Ans.*

3. Divide the number a into two such parts that the product of the mth power of one and the nth power of the other shall be a maximum.
In the ratio $m:n$. *Ans.*

24. The subject of Maxima and Minima is one of the most important in the applications of the Calculus to Geometry, Mechanics, etc. It is often necessary to derive the expression for the function to be investigated, and in testing this, attention should be paid to the remark in Example 1 (f) of the preceding exercise.

EXERCISE 13

1. A box with a square base and open top is to be constructed to contain 108 cubic inches. What must be its dimensions to require the least material ?* Base 6 inches square, height 3 inches. *Ans.*

2. The strength of a rectangular beam varies as the product of the breadth b and the square of the depth d. What are the dimensions of the strongest beam that can be cut from a log whose cross section is a circle a inches in diameter ?† Breadth is $\frac{1}{3}a\sqrt{3}$ inches. *Ans.*

$f(x)=\sqrt{u}$, $f'(x)=\frac{1}{2\sqrt{u}}\frac{du}{dx}$, and $f''(x)\equiv-\frac{1}{4\sqrt{u^3}}\frac{du}{dx}+\frac{1}{2\sqrt{u}}\frac{d^2u}{dx^2}$, so that $f'(x)$ vanishes only if $\frac{du}{dx}=0$, and then $f''(x)$ has the same sign as $\frac{d^2u}{dx^2}$.

* HINT. Let x be the side of the base, y the height, then $x^2y=108$, *i.e.* $y=\frac{108}{x^2}$; and since the material is x^2+4xy, we find by substituting for y the function $x^2+\frac{432}{x}$.

† HINT. The strength therefore equals bd^2 multiplied by some constant, which may be dropped by the remark of § 24. But $d^2=a^2-b^2$; hence the function is $b(a^2-b^2)$, b being the variable.

3. Find the dimensions of the stiffest beam that can be cut from the same log as in 2, given that the stiffness varies as the product of the breadth and the cube of the depth. Breadth $\frac{1}{2}a$ inches. *Ans.*

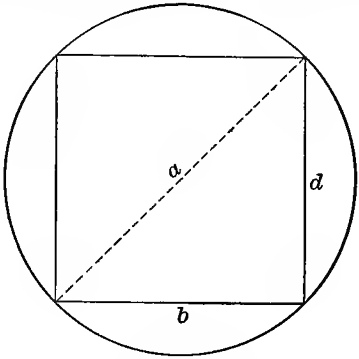

FIG. 15

4. The equation of the path of a projectile (see Fig. 16) is

$$y = x \tan \alpha - \frac{gx^2}{2\,v_0^2 \cos^2 \alpha},$$

where α is the angle of elevation and v_0 the initial velocity. Find the greatest height. $\frac{v_0^2 \sin 2\alpha}{2g}$. *Ans.*

FIG. 16

5. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. *Ans.* Sides are $a\sqrt{2}$ and $b\sqrt{2}$.

6. Find the altitude of the right cylinder of greatest volume inscribed in a sphere of radius r. Altitude $= \frac{2r}{\sqrt{3}}$. *Ans.*

7. Assuming that the brightness of the illumination of a surface varies directly as the sine of the angle under which the light strikes the surface and inversely as the square of the distance from the source of light, find the height of a light placed directly over the center of a circle of radius a when the illumination of the circumference is greatest.

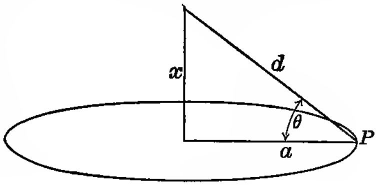

FIG. 17

From Fig. 17, the brightness at P is given by

$$\frac{\kappa \sin\theta}{d^2} = \frac{\kappa x}{d^3} = \kappa \frac{x}{\sqrt{(a^2+x^2)^3}} = \kappa\left(\frac{x^2}{(a^2+x^2)^3}\right)^{\frac{1}{2}}.$$

Hence the brightness is a maximum when $\dfrac{x^2}{(a^2+x^2)^3}$ is a maximum.

$$x = \frac{a}{\sqrt{2}}. \quad \textit{Ans.}$$

25. Expansion of Functions. By actual division

$$(19) \qquad \frac{1}{1-x} = 1 + x + x^3 + \cdots + x^n + \left(\frac{1}{1-x}\right)x^{n+1},$$

where n is some positive integer. In this simple way we may find for the function $\dfrac{1}{1-x}$ *an equivalent polynomial all of whose coefficients save that of* x^{n+1} *are constants.* By transposition (19) becomes

$$(20) \qquad \frac{1}{1-x} - (1 + x + x^2 + x^3 + \cdots + x^n) = \frac{1}{1-x}x^{n+1}.$$

Now let x be some number *numerically less than* 1, say $x = .5$, and suppose we wish the value of $\dfrac{1}{1-x}$ correct within one one-hundredth, *i.e.* correct to two decimal places. Let us then determine for what values of n the term $\dfrac{1}{1-x}x^{n+1}$ when $x = .5$ is less than .01, *i.e.* solve the inequality $\dfrac{1}{1-.5}.5^{n+1} < .01$. We find $n > 6$.

Furthermore, if x is numerically less than .5, $\frac{1}{1-x}$ and x^{n+1} are less than for $x = .5$, so that taking $n = 7$ (*i.e.* >6), $\frac{1}{1-x}x^8 < .01$ for every value of x not numerically greater than .5. And we now see from (20) that the function $\frac{1}{1-x}$ may be replaced by the polynomial $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$ for all values of x numerically equal to or less than .5 if results correct only to hundredths' place are desired.

Precisely the same reasoning holds for *any* value of x numerically less than unity, since for any such value x^{n+1} can be made as small as we please by taking n sufficiently great. But this reasoning does not hold for any value equal to or exceeding 1 numerically. We may then state this theorem:

For any value of x numerically less than unity, the function $\frac{1}{1-x}$ may be represented with any desired degree of accuracy by a sufficiently great number of terms of the polynomial,

$$1 + x + x^2 + x^3 + \cdots.$$

The Differential Calculus enables us to obtain a similar theorem for many other functions, as will now be explained. In all practical computations results correct to a certain number of decimal places are sought, and since the process in question replaces a function perhaps difficult to calculate by a polynomial with constant coefficients, it is therefore of great practical importance in simplifying such computations.

26. Theorem of the Mean. If $f(x)$ and $f'(x)$ are continuous as x varies from a to b, then there is at least one value of x, say x_1, between a and b, such that

$$\frac{f(b)-f(a)}{b-a}=f'(x_1). \tag{21}$$

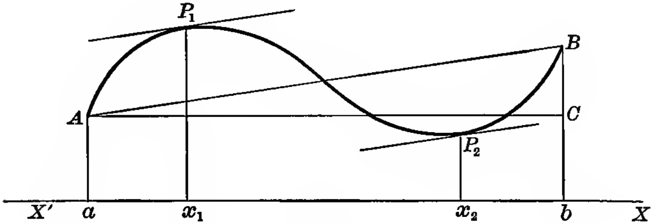

FIG. 18

In Fig. 11, $f(b)-f(a)=CB,\ \ b-a=AC,$

$\therefore\ \dfrac{f(b)-f(a)}{b-a}=$ slope of AB, and at *each* of the points P_1 and P_2 the tangent is parallel to AB, and hence (21) is true if x_1 is the abscissa of P_1 or P_2.*

Multiplying (21) out gives

$$f(b)=f(a)+(b-a)f'(x_1), \tag{22}$$

where it must be remembered $a>x_1>b$.

A more general theorem than (21) is enunciated as follows:

If $f(x)$ and the $(n+1)$ successive derivatives $f'(x)$, $f''(x)$, $\cdots$, $f^{(n+1)}(x)$ are continuous when x varies from a

* This proof of the theorem of the mean is not mathematically rigorous, but merely illuminates the significance of (21). The student should draw other figures, and especially such that the necessary conditions of the continuity of $f(x)$ and $f'(x)$ fail.

to b, then there is at least one value of x, say x_1, between a and b such that

$$(23) \qquad f(b) = f(a) + \frac{(b-a)}{1} f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \frac{(b-a)^3}{\lfloor 3} f'''(a) + \cdots + \frac{(b-a)^n}{\lfloor n} f^{(n)}(a) + \frac{(b-a)^{n+1}}{\lfloor n+1} f^{(n+1)}(x_1).$$

The proof of (23) is beyond the scope of this book.* The student should, however, carefully note the law by which the expression on the right is constructed.

Putting for b in (23) the variable x, we get *Taylor's Theorem*,

$$(24) \qquad f(x) = f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^{n+1}}{\lfloor n+1} f^{(n+1)}(x_1), \text{ where } a < x_1 < x.$$

Finally, setting $a = 0$ in (24), we find *Maclauren's Theorem*,

$$(25) \qquad f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^n}{\lfloor n} f^{(n)}(0) + \frac{x^{n+1}}{\lfloor n+1} f^{(n+1)}(x_1) \text{ where, } 0 < x_1 < x.$$

If in (23) we put $b = a + x$, we obtain another form of Taylor's theorem,

$$f(a+x) = f(a) + \frac{x}{1} f'(a) + \frac{x^2}{\lfloor 2} f''(a) + \cdots \text{ etc.}$$

This formula (25) gives $f(x)$ in the form of a polynomial in x with constant coefficients save that of x^{n+1}, which, since x_1 lies between 0 and x, is a function of x; that is,

* An excellent discussion is given in Gibson's *An Elementary Treatise on the Calculus*, London, 1901, p. 390.

we have the generalization of the example of § 25 as follows:

A function $f(x)$ for certain values of the variable may be represented with any desired degree of accuracy by the polynomial,*

$$f(0)+\frac{x}{1}f'(0)+\frac{x^2}{\underline{|2}}f''(0)+\frac{x^3}{\underline{|3}}f'''(0)+\cdots+\frac{x^n}{\underline{|n}}f^{(n)}(0).$$

By "expansion of a function" is meant the forming of this polynomial. Of course n is indefinite, and must be taken great enough to give the desired degree of accuracy. It is of greatest *theoretical* importance to determine for what values of x the polynomial represents the function when n is taken indefinitely great. This consists in examining for what values of x

$$\text{Limit}\left(\frac{x^{n+1}}{\underline{|n+1}}f^{(n+1)}(x_1)\right)_{n=\infty}=0,$$

for this term is the *difference* between the function and the polynomial.

EXERCISE 14

1. Expand $\sin x$.

Since $f(x)=\sin x$, and for $x=0$, $f(0)=\sin 0=0$;

then $f'(x)=\cos x$, and for $x=0$, $f'(0)=1$;

$f''(x)=-\sin x$, $f''(0)=0$;

$f'''(x)=-\cos x$, $f'''(0)=-1$;

$f^{iv}(x)=\sin x$, $f^{iv}(0)=0$;

etc. etc.

Hence $$\sin x = x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\frac{x^9}{\underline{|9}}-\text{etc.}$$

2. Show that the expansion of $\cos x$ is

$$\cos x = 1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\frac{x^8}{\underline{|8}}-\text{etc.}$$

* Namely, for all values of x such that the "remainder" $\frac{x^{n+1}}{\underline{|n+1}}f^{(n+1)}(x_1)$ is less than the limit of error. This question is often difficult to settle.

3. Expand e^x.

Since $f(x) = e^x$, and all its derivatives are likewise e^x, while $e^0 = 1$, we obtain

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 5} + \cdots.$$

Putting $x = 1$, we find

$$e = 1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots,$$

the expression given in § 10.

The expansions of $\sin x$, $\cos x$, and e^x are remarkable in that they hold for *every* value of x, positive and negative.

4. Prove the following expansions:

(*a*) $\log_a (1 + x) = \log_a e \left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right).$

(*b*) $(1 + x)^m = 1 + mx + \frac{m(m-1)}{\lfloor 2} x^2 + \frac{m(m-1)(m-2)}{\lfloor 3} x^3 + \cdots.$

(*b*) is the *binomial formula*. These expansions hold only *for values of x numerically less than* 1.

Taylor's Theorem (24) differs from (25) in that we are to consider values of the variable x near some given number a, since (24) is a polynomial in $(x - a)$ in the same sense that (25) is a polynomial in x. It is evident that no greater difficulty arises in the application of (24) to a given function than has been already pointed out.

27. Differentials. From (23) we are able to find an expansion for the increment of a function in powers of the increment of the variable as follows:

Write $b = x + \Delta x$, $a = x$, $\therefore b - a = \Delta x$, and (23) becomes, after transposing $f(x)$,

$$(26) \quad f(x + \Delta x) - f(x) = \Delta x f'(x) + \frac{(\Delta x)^2}{\lfloor 2} f''(x) + \cdots,$$

or $$(27) \qquad \Delta f(x) = f'(x)\Delta x + f''(x)\frac{(\Delta x)^2}{\lfloor 2} + \cdots.$$

Now, if we suppose Δx to diminish toward zero, the first term $f'(x)\Delta x$ of the right-hand member will ultimately

greatly exceed the sum of the remaining terms, since these contain higher powers of Δx. For this reason $f'(x)\Delta x$ is called the *principal part of the increment of* $f(x)$. Also, when we wish to emphasize the fact that the variable Δx is to approach zero as a limit, we write dx, called *differential* x, instead of Δx, and the principal part of the increment $f'(x)dx$ we call the *differential of the function;* that is,

$$df(x) = f'(x)\,dx. \tag{28}$$

The following definitions are fundamental:

A differential (or infinitesimal) is a variable whose limit is zero.

The differential of the independent variable is an increment of that variable whose limit is zero.

The differential of the dependent variable is the principal part of the increment of that variable, and equals the product of the derivative and the differential of the independent variable (28).

From (28), we see that if y is a function of x, then

$$dy = \frac{dy}{dx}\,dx. \tag{29}$$

EXERCISE 15

1. Prove by (28) and (29) the following differentials:

(*a*) $d(3x^2) = 6x\,dx.$

(*b*) $d\log_e x = \dfrac{dx}{x}.$

(*c*) $de^x = e^x dx.$

(*d*) $dx^m = mx^{m-1}dx.$

(*e*) $d\sqrt{1-x} = -\dfrac{dx}{2\sqrt{1-x}}.$

(*f*) $d\sin 2x = 2\cos 2x\,dx.$

(*g*) $d\tan\left(\dfrac{1}{x}\right) = -\dfrac{\sec^2\left(\dfrac{1}{x}\right)}{x^2}dx.$

(*h*) If $y = x\log_e x$, then $dy = (1 + \log_e x)\,dx.$

2. If $y = uv$, then

$$dy = \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx = u\frac{dv}{dx}dx + v\frac{du}{dx}dx, \text{ or } dy = u\,dv + v\,du.$$

3. Show that $$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}.$$

4. State the rules I–V for differentiation in terms of differentials instead of derivatives.

28. We may write (27) after replacing Δx by dx,

$$(30)\quad \Delta f(x)=f'(x)dx+dx^2\left(\frac{f''(x)}{\underline{|2}}+\frac{f'''(x)}{\underline{|3}}dx+\cdots\right).$$

Now, since by (28) $f'(x)dx$ is the differential of the function, (30) shows that $\Delta f(x)$ and $df(x)$ differ by a term containing the factor dx^2. Such a quantity is called a *differential of the second order;* in general, any quantity containing as a factor the *product of two differentials* is thus designated.

The increment of a function differs from its differential by a differential of the second order.

EXAMPLES

1. *Differential of a product uv.*

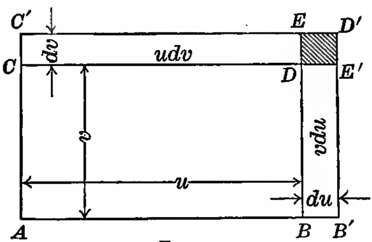

FIG. 19

Let $u=AB$, $v=AC$, then $uv=$ area $ABCD$. If $du=BB'$, $dv=CC'$, then

$$\begin{aligned}\Delta(uv)&=\text{area } AB'C'D'-\text{area } ABCD\\&=\text{area } CDC'E+\text{area } BB'DE'+\text{area } DE'ED'\\&=u\,dv+v\,du+du\cdot dv.\end{aligned}$$

Now $du\cdot dv$ is a differential of the second order, $\therefore$ *principal part* of $\Delta(uv)$ is $u\,dv+v\,du$; *i.e.* $d(uv)=u\,dv+v\,du$. (Cf. Ex. 2, § 27.)

2. *Differential of an area.*

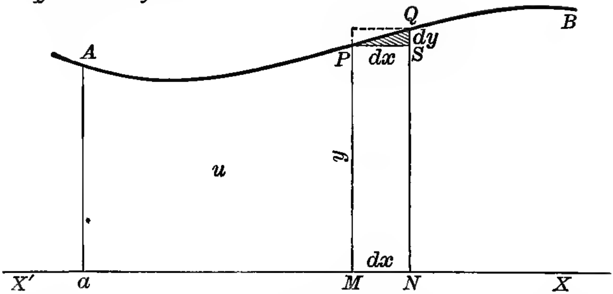

FIG. 20

Consider the area $aAPM$ bounded by any curve, the axis XX' and the ordinates aA, MP, and call this area u. Then if $MN = dx$, Δu $=$ area $aAQN -$ area $aAPM =$ area $MPQN$. $\therefore \Delta u = y\,dx +$ area PSQ. But area $PSQ < dx \cdot dy$. $\therefore PSQ = k\,dx\,dy$, where k is some number < 1. Hence area PSQ is a differential of the second order, and $\therefore du = y\,dx$.

The differential of the area bounded by any curve, the axis XX', and two ordinates is the product of the ordinate of the curve and the differential of the abscissa.

3. *Differential of the volume of a solid of revolution.*

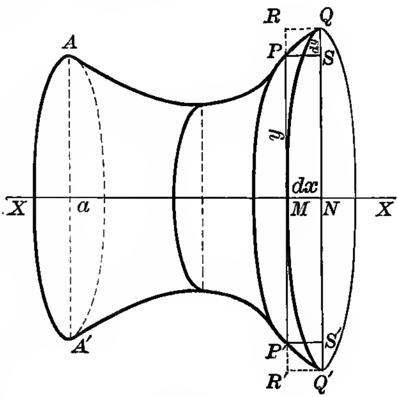

FIG. 21

Let the solid be generated by revolving a curve APQ around XX', and denote the volume $APA'P'$ by v. If $dx = MN$, then $\Delta v =$ volume $AQA'Q'$ $-$ volume $APA'P'$, or $\Delta v =$ volume of the cylinder $PSP'S'$ $+$ volume generated by the curvilinear $\Delta\,PSQ$. Now the volume of the cylinder $PSP'S' = \pi y^2 dx$, since y $= PM =$ radius of base and $dx =$ altitude. The volume generated by the curvilinear $\Delta\,PSQ <$ volume generated by the rectangle $PRSQ$, and this last volume $= \pi\overline{NQ}^2 \cdot MN - \pi\overline{MP}^2 \cdot MN = \pi(2y\,dy + dy^2)dx$.

We see therefore that $\Delta v = \pi y^2 dx +$ a differential of the second order, *i.e.* $dv = \pi y^2 dx$.

The differential of the volume of a solid of revolution generated by revolving any curve around the axis XX' equals π times the product of the square of the ordinate and the differential of the abscissa.

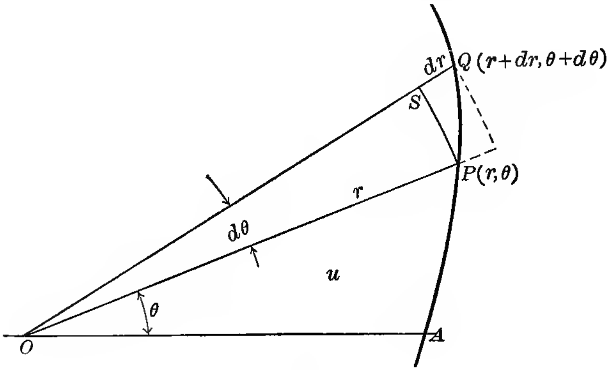

FIG. 22

4. Show that the differential of the area u bounded by a curve AP and two radii vectores OA and OP is given by $du = \frac{1}{2} r^2 d\theta$, where (r, θ) are the polar coördinates of P.

CHAPTER IV

INTEGRATION

29. Indefinite Integral. *Integration* consists in finding a function of which a given *differential* expression, such as $x\,dx$, $\sin x\,dx$, $\frac{du}{u}$, etc., is the differential. The function thus found is called the *integral* of the given differential expression, and the operation is indicated by prefixing the integral sign $\int$. Thus, since

$$d(\tfrac{1}{2}x^2) = x\,dx, \qquad \therefore \int x\,dx = \tfrac{1}{2}x^2;$$

$$\int dx = x, \qquad \int \sin x\,dx = -\cos x, \text{ etc.}$$

In general,

$$\int f(x)\,dx$$

means to find a function $F(x)$ such that

$$dF(x) = f(x)\,dx,$$

i.e.

$$f(x) = \frac{d}{dx}F(x).$$

Constant of Integration. Since $d(\frac{1}{2}x^2 + C)$ also equals $x\,dx$, *no matter what the constant* C is, we have

$$\int x\,dx = \tfrac{1}{2}x^2 + C,$$

where C is any constant whatever, called the *constant of integration.* We see, therefore, that a given differential

expression may have infinitely many integrals, found by giving to the constant of integration different values. Thus

$$\int f(x)\,dx = F(x) + C,$$

and since C is unknown and *indefinite*, $F(x) + C$ is called the *indefinite* integral of $f(x)\,dx$.

Of course, the same differential expression has an indefinite number of *distinct* integrals, but what has just been said shows that the difference of any two of these must be a constant.

30. Rules for Integration. From Rule V in differentiation, if v is any function of x, and κ a constant, then

$$\frac{d}{dx}(\kappa v) = \kappa\frac{dv}{dx}, \quad i.e.\ d(\kappa v) = \kappa\, dv.$$

Integrating, we have, since if two differential expressions are equal so are their integrals equal,

$$\int \kappa\, dv = \int d(\kappa v),$$

or, since $$\int d(\kappa v) = \kappa v,$$

$$\kappa v = \int \kappa\, dv.$$

But $$\kappa \int dv = \kappa v.$$

(31) $$\therefore \int \kappa\, dv = \kappa \int dv.$$

XXIII. A constant factor may be written either before or after the integral sign.

The chief application of XXIII is to be found in cases like the following:

To work out $\int x\,dx$. If we multiply $x\,dx$ by 2, we have an exact differential, since

$$d(x^2) = 2x\,dx,$$

$$\therefore \int 2x\,dx = x^2;$$

but by XXIII,

$$\int 2x\,dx = 2\int x\,dx,$$

$$\therefore \int x\,dx = \frac{x^2}{2}.$$

From (31) we may also write

$$\int f(x)\,dx = \frac{1}{\kappa}\int \kappa f(x)\,dx. \tag{32}$$

Integral of a Sum of Differential Expressions. If u and v are functions of x, then

$$d(u+v) = \frac{d}{dx}(u+v)\,dx = du + dv.$$

$$\therefore \int (du + dv) = \int d(u+v) = u + v = \int du + \int dv.$$

This result gives Rule

XXIV. The integral of any algebraic sum of differential expressions equals the same algebraic sum of the integrals of these expressions taken separately.

That is, *e.g.*,

$$\int (x+3)\,dx = \int (x\,dx + 3\,dx) = \int x\,dx + \int 3\,dx = \tfrac{1}{2}x^2 + 3x + C.$$

31. From any result in differentiation may always be derived an integration formula, and we now proceed to obtain some of the simpler ones, making use of § 18.

Since by VIII,

$$d(v^{m+1}) = (m+1)v^m\,dv,$$

then, integrating,

$$v^{m+1} = \int (m+1)v^m\,dv = (m+1)\int v^m\,dv. \quad \text{(XXIII)}$$

(33) $$\therefore \int v^m\,dv = \frac{v^{m+1}}{m+1}.$$

From IX, $$d\log_e v = \frac{dv}{v},$$

(34) $$\therefore \int \frac{dv}{v} = \log_e v.$$

In the same way we might go through with each formula in § 18. It will suffice for our purpose to tabulate a few of the results:

XXV. $$\int v^m\,dv = \frac{v^{m+1}}{m+1} + C\ (m \neq -1).$$

XXVI. $$\int \frac{dv}{v} = \log_e v + C.$$

XXVII. $$\int a^v\,dv = \frac{a^v}{\log_e a} + C.$$

XXVIII. $$\int \sin v\,dv = -\cos v + C.$$

XXIX. $$\int \cos v\,dv = \sin v + C.$$

XXX. $$\int \frac{dv}{\sqrt{a^2 - v^2}} = \text{arc sin}\,\frac{v}{a} + C.$$

XXXI. $$\int \frac{dv}{a^2 + v^2} = \frac{1}{a}\,\text{arc tan}\,\frac{v}{a} + C.$$

EXAMPLES

1. Find $$\int \frac{dx}{\sqrt{1-x}}.$$

This is the same thing as $\int (1-x)^{-\frac{1}{2}}\,dx$, which resembles XXV. For put

$$1 - x = v, \text{ then } -dx = dv, \text{ or } dx = -dv.$$

$$\therefore \int (1-x)^{-\frac{1}{2}}dx = \int v^{-\frac{1}{2}} - dv = -\int v^{-\frac{1}{2}}dv.$$

$\therefore$ by XXV $$\int v^{-\frac{1}{2}}dv = \frac{v^{\frac{1}{2}}}{\frac{1}{2}} + C,$$
and by substituting again,

$$\int \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x} + C.$$

2. Work out $$\int \frac{3\,ax\,dx}{c^2 - b^2x^2}.$$

Taking out constant factor $3\,a$ (XXIII), this becomes

$$3\,a\int \frac{x\,dx}{c^2 - b^2x^2},$$
and this resembles XXVI.

For put $c^2 - b^2x^2 = v$, $\therefore\ -2\,b^2x\,dx = dv$, or $x\,dx = -\dfrac{dv}{2\,b^2}$.

$$\therefore\ 3\,a\int \frac{x\,dx}{c^2 - b^2x^2} = 3\,a\int \frac{-\frac{dv}{2\,b^2}}{v} = -\frac{3\,a}{2\,b^2}\int \frac{dv}{v} = -\frac{3\,a}{2\,b^2}\log v + C.$$

$$\therefore\ \int \frac{3\,ax\,dx}{c^2 - b^2x^2} = -\frac{3\,a}{2\,b^2}\log(c^2 - b^2x^2) + C.$$

3. Find $$\int \frac{dx}{9 + 4\,x^2}.$$

This resembles XXXI, if $a = 3$, $2\,x = v$.

Then $2\,dx = dv$, and since the given integral by (32) is the same as

$$\frac{1}{2}\int \frac{2\,dx}{3^2 + (2\,x)^2} \text{ or } \frac{1}{2}\int \frac{dv}{a^2 + v^2},$$

we find by XXXI, $$\int \frac{dx}{9 + 4x^2} = \frac{1}{6}\text{arc tan}\,\frac{2\,x}{3} + C.$$

By studying the above examples the student will see that integration depends upon comparison of the given integral with certain standard forms. To be able to tell quickly what form the given integral resembles is absolutely essential.

Tables of standard forms* have been constructed containing all integrals occurring in ordinary work.

EXERCISE 16

1. Prove the following integrations:

(*a*) $\int (ax + bx^2)\, dx = \frac{1}{2}ax^2 + \frac{1}{3}bx^3 + C.$ (Use XXIV.)

(*b*) $\int \frac{\sin x\, dx}{1 - \cos x} = \log_e (1 - \cos x) + C.$

(*c*) $\int \sqrt{a^2 - x^2}\, x\, dx = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C.$ (Use XXV, $v = a^2 - x^2$).

(*d*) $\int \sin (2x)\, dx = -\frac{1}{2}\cos 2x + C.$

(*e*) $\int e^{-x}\, dx = -e^{-x} + C.$

(*f*) $\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\arcsin (2x) + C.$

(*g*) $\int \frac{x\, dx}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2} + C.$

(*h*) $\int \frac{dx}{1 - x} = -\log_e (1 - x) + C.$

(*i*) $\int \sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}} + C;\ \int \frac{dx}{x^3} = -\frac{1}{2x^2} + C.$

(*j*) $\int \tan x\, dx = -\log_e \cos x + C.$ (Put $\tan x = \frac{\sin x}{\cos x}$ and use XXVI.)

(*k*) $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$ (Put $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.)

2. *Special Devices in Integration.*

(*a*) *By partial fractions,* when we have to integrate a rational fraction times dx, and this fraction can be replaced by partial fractions.

* *E.g.*, B. O. Peirce's *A Short Table of Integrals*, Ginn & Co., 1899.

For example, $$\int \frac{dx}{a^2 - x^2}.$$

Putting $$\frac{1}{a^2 - x^2} = \frac{A}{a - x} + \frac{B}{a + x},$$
and clearing of fractions,
$$1 = x(A - B) + a(A + B).$$
$$\therefore\ A - B = 0,\quad a(A + B) = 1,\ \text{ or }\ A = B = \frac{1}{2a}.$$
$$\therefore\ \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{dx}{a - x} + \frac{1}{2a}\int \frac{dx}{a + x} = \frac{1}{2a}(\log_e(a+x) - \log_e(a-x))$$
$$= \frac{1}{2a}\log_e\left(\frac{a + x}{a - x}\right) + C.$$

(*b*) *By change of variable.*

Find $\int \sqrt{a^2 - x^2}\,dx$. Substitute $x = a\cos\theta$;
$$\therefore\ dx = -a\sin\theta\,d\theta,\quad \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\cos^2\theta} = a\sin\theta,$$
and $$\int \sqrt{a^2 - x^2}\,dx = -a^2\int \sin^2\theta\,d\theta = -\frac{a^2}{2}\theta + \frac{a^2}{4}\sin 2\theta + C$$
by Ex. 1 (*k*). Now
$$\theta = \text{arc}\cos\frac{x}{a},\quad \sin 2\theta = 2\sin\theta\cos\theta = 2\sqrt{1 - \frac{x^2}{a^2}}\cdot\frac{x}{a}.$$
$$\therefore\ \int \sqrt{a^2 - x^2}\,dx = -\frac{a^2}{2}\text{arc}\cos\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2}.$$

3. Prove $$\int \frac{dx}{x^2 + 3x} = \log_e\sqrt[3]{\frac{x}{x + 3}} + C.$$

The following two examples illustrate the manner of determination of the constant of integration by means of so-called *initial conditions*.

4. Find the amount of a sum of money increasing *continuously* at compound interest of r per cent.

We found, page 43, that, in derivatives, P being the sum sought,
$$\frac{dP}{dt} = \frac{r}{100}P.$$

Multiplying by dt and dividing by P, we have
$$\frac{dP}{P} = \frac{r}{100}dt;$$

integrating, $$(1)\quad \log_e P = \frac{r}{100}t + C.$$

Let now a equal the *initial* sum of money; that is, the sum started with, so that $P = a$ when $t = 0$; substituting these in this equation, we have $\log_e a = C$, so that (1) becomes $\log_e P = \frac{r}{100}t + \log_e a$, or, transposing,

$$\log_e P - \log_e a = \frac{r}{100}t, \text{ or } \log_e\left(\frac{P}{a}\right) = \frac{r}{100}t;$$

i.e.
$$P = ae^{\frac{r}{100}t}. \quad \textit{Ans.}$$

5. Find the relation between s (space) and t (time) for uniformly accelerated rectilinear motion.

Since the acceleration $\frac{dv}{dt}$ is constant, say f, we have $\frac{dv}{dt} = f$.

Multiplying by dt, $dv = f\,dt$, and integrating, $v = ft + C$.

To determine C, let the *initial* velocity be v_0, *i.e.* $v = v_0$ when $t = 0$, or $v_0 = 0 + C$. $\therefore\ v = ft + v_0$.

Since $v = \frac{ds}{dt}$, $\therefore\ \frac{ds}{dt} = ft + v_0$, and multiplying by dt, $ds = ft\,dt + v_0 dt$.

Integrating, $s = \frac{1}{2}ft^2 + v_0 t + C$, and if $s = s_0$ when $t = 0$, we have finally $s = \frac{1}{2}ft^2 + v_0 t + s_0$. *Ans.*

32. Definite Integral. We have already seen that the indefinite integral contains an arbitrary constant, *the constant of integration*, and has for that reason an indefinite value. By making suitable assumptions, now to be explained, we are able to dispose of this inconvenience.

In § 28, Example 2, it was shown that the differential of the area u between a curve $MABC$, the axis XX', and any two ordinates was given by

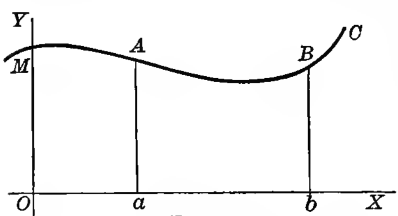

FIG. 23

$$du = y\,dx.$$

$$\therefore\ u = \int y\,dx + C.$$

Here, of course, y is some function of x determined from the equation of the curve, and $\therefore \int y\,dx =$ some function of x, say $F(x)$.

$$\therefore\ u = F(x) + C.$$

Let us now agree to reckon the area from the axis YY', so that when $x=a$, $u=$ area $OaAM$, etc.

Under this assumption, when $x=0$, $u=0$, and

$$\therefore\ 0=F(0)+C, \text{ or } C=-F(0),$$

and we have

$$u=F(x)-F(0).$$

Now area $OaAM=F(a)-F(0)$.

Area $ObBM=F(b)-F(0)$. Subtracting, we have

Area $abAB\ =F(b)-F(a)$,

or,

The difference of values of the $\int y\,dx$ *for* $x=b$ *and* $x=a$ *gives the area bounded by the curve whose ordinate is* y, *the axis* XX', *and the ordinates at* a *and* b.

This difference is represented by the symbol

$$\int_a^b y\,dx, \tag{35}$$

read, "integral from a to b of $y\,dx$"; the operation is called *integration between limits*, a being the *lower*, b the *upper* limit.

We see therefore that (35) or, what is the same thing,

$$\int_a^b f(x)\,dx \tag{36}$$

always has a *definite* value, and is accordingly a *definite integral.* For if

$$\int f(x)dx=F(x)+C, \text{ then} \tag{37}$$

$$\int_a^b f(x)dx=F(b)+C-(F(a)+C)=F(b)-F(a), \tag{38}$$

and the *constant of integration has disappeared.*

33. Areas of Plane Curves. From § 32, we have the theorem: *Given any plane curve $y=f(x)$, the definite integral $\int_a^b f(x)dx$ gives the area bounded by that curve, the axis XX' and the ordinates at a and b.*

To find the area bounded by two given curves, we get the area between each and XX' and then subtract.

Volumes of solids of revolution.

Precisely as in § 32 and remembering the result of Example 3, § 28 we prove that:

Given any plane curve $y=f(x)$, the definite integral $\int_a^b \pi y^2 dx$ gives the volume generated by revolving around XX' the portion of the curve between the ordinates at a and b.

The two theorems just given find numerous applications in Geometry.

EXERCISE 17

1. Find the area of the curve $y=x^2-9$ lying below XX'.

Here $\int y\,dx = \int (x^2-9)\,dx$, and since for $y=0$, $x=\pm 3$, the limits are $+3$ and -3, *i.e.* area $=\int_{-3}^{3}(x^2-9)\,dx$. 36. *Ans.*

2. Find the area of the circle $x^2+y^2=a^2$.

Since $y=\sqrt{a^2-x^2}$, $\int y\,dx = \int \sqrt{a^2-x^2}\,dx$ which has been worked out in Exercise 16, Example 2 (b). For the semicircle the limits are $+a$ and $-a$.

3. Show that the area of the ellipse $b^2x^2+a^2y^2=a^2b^2$ is to the area of the circle whose diameter is the major axis $2a$ as $b:a$.

4. Find the area of one arch of sine curve $y=\sin x$. 2. *Ans.*

5. Find area between the equilateral hyperbola $xy=1$, the axis XX', and the ordinates at $x=a$, $x=b$. $\log_e\left(\frac{b}{a}\right)$. *Ans.*

6. Find the volume of the sphere.

Since we have to revolve the circle $x^2+y^2=a^2$, or $y^2=a^2-x^2$ around XX', then $\int \pi y^2 dx = \pi \int (a^2-x^2)\,dx$. The limits are $+a$ and $-a$. $\frac{4}{3}\pi a^3$. *Ans.*

7. Find the volume generated by revolving around XX' the parabola $y^2=4x$ and cut off by a plane perpendicular to XX' at the distance of 4 to the right of the origin. 32. *Ans.*

34. Definite Integral as the Limit of a Sum of Differential Expressions. In the Differential Calculus the student was asked to bear in mind that everything was built up from a fundamental limit, *the limit of a quotient whose denominator approached zero.* We are now to see that the definite integral is the *limit of a sum of differential expressions.*

If
$$\int f(x)dx = F(x)+C,$$
then
$$\frac{d}{dx}F(x)=f(x) \text{ and } \int_a^b f(x)dx = F(b)-F(a)$$
gives the area bounded by the curve $y=f(x)$ (Fig. 24), the axis XX', and the ordinates at $x=a$, $x=b$.

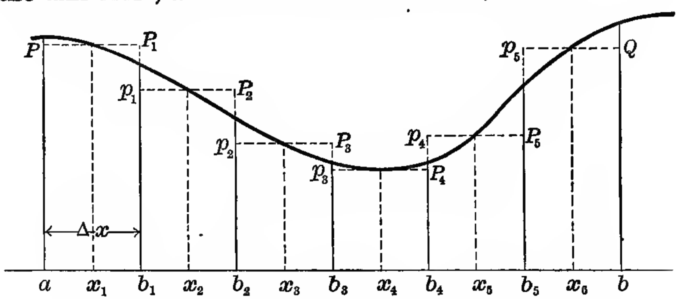

FIG. 24

Now divide the segment ab into any number of equal parts, say 6, $a_1b_1=b_1b_2=\cdots=b_5b$, and call the length of each division Δx. Erect the ordinates at these points, and

apply the theorem of the mean (§ 26) to each division. In the present case $F(x)$ takes the place of $f(x)$ in (21), and $f(x)$ replaces $f'(x)$; for the first interval ab_1, $a=a$, $b=b_1$, and x_1, lying between a and b, is marked in the figure. Draw the ordinate of x_1. Then (21) gives

$$\frac{F(b_1)-F(a)}{b_1-a}=f(x_1),$$

or, since $$b_1-a=\Delta x,$$

(39) $$F(b_1)-F(a)=f(x_1)\Delta x.$$

In the same way (21) applied to each of the remaining five segments gives the equations

(40) $$\begin{cases} F(b_2)-F(b_1)=f(x_2)\Delta x, \\ F(b_3)-F(b_2)=f(x_3)\Delta x, \\ F(b_4)-F(b_3)=f(x_4)\Delta x, \\ F(b_5)-F(b_4)=f(x_5)\Delta x, \\ F(b)\ \ -F(b_5)=f(x_6)\Delta x. \end{cases}$$

Adding the six equations (39) and (40), we find

(41) $$F(b)-F(a)=f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x \\ +f(x_4)\Delta x+f(x_5)\Delta x+f(x_6)\Delta x.$$

But $f(x_1)\Delta x=$ area of the rectangle aPP_1b_1,

$f(x_2)\Delta x=$ area of the rectangle $b_1p_1P_2b_2$,

etc.,

so that the sum on the right equals the area

$$aPP_1p_1P_2p_2P_3p_3P_4p_4P_5p_5Qb;\ \textit{i.e.}$$

(42) $F(b)-F(a)=$ area between the broken line

$PP_1p_1\cdots p_5Q$ and XX',

and this is true independently of the number of parts into which ab is divided. Hence for *any* number n of equal parts

$$(43)\quad F(b)-F(a)=f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_n)\Delta x,$$

$$(44)\qquad \text{and } \Delta x=\frac{b-a}{n}.$$

Equations (43) and (44) hold when n increases without limit, and then Δx becomes dx (§ 27), *i.e.* a variable whose limit is zero.

$$\therefore\ F(b)-F(a)=\underset{n=\infty}{\text{Limit}}\,(f(x_1)dx+f(x_2)dx+\cdots+f(x_n)dx),$$

or, by (38),

$$(45)\quad \int_a^b f(x)dx=\underset{n=\infty}{\text{Limit}}\,(f(x_1)dx+f(x_2)dx+\cdots+f(x_n)dx).$$

And now we see very clearly why $\int_a^b f(x)dx$ gives the area under the curve, for as n increases, the broken line $PP_1p_1P_2p_2\cdots p_5Q$ approaches the curve itself, and the sum $f(x_1)dx+\cdots+f(x_n)dx$ always represents the area under this broken line.

Integrating between limits is accordingly spoken of as "summing up"; the integration sign $\int$ is historically a distorted S, the first letter of *sum*. But let the student not forget that the definite integral is not a sum, but the *limit of a sum, the number of terms increasing without limit, and each term itself diminishing toward zero.*

The problem of finding the area is then to be thought of thus: Divide the interval on xx' into any number of equal parts, and at a point within each division erect an ordinate to the curve; construct the rectangles on the divisions as bases, with the corresponding ordinate as

altitude. Then finding the area consists in summing up these rectangles and taking the limit of this sum as the number of divisions increases without limit.

As an example of the great number of problems in Physics and other branches of Mathematics which involve in their solution definite integrals, consider the following:

To determine the amount of attraction exerted by a thin, straight, homogeneous rod of uniform thickness and of length l upon a material point P of mass m, situated in the line of direction of the rod.

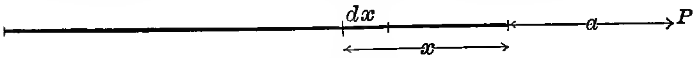

FIG. 25

Imagine the rod (see Fig. 25) divided up into equal infinitesimal portions (elements) of length dx. If M = mass of rod, then

$$\frac{M}{l}dx = \text{mass of any element.}$$

The law of attraction being Newton's Law, *i.e.* attraction = product of masses ÷ square of distance, then

$$\text{attraction of element } dx \text{ on } P = \frac{\frac{M}{l}m\,dx}{(x+a)^2},$$

and the total attraction is the *sum* of these from $x = 0$ to $x = l$.

$$\therefore \text{ Force} = \int_0^l \frac{\frac{M}{l}m\,dx}{(x+a)^2} = \frac{Mm}{l}\int_0^l \frac{dx}{(x+a)^2},$$

or integrating, $\quad \text{Force} = \dfrac{Mm}{l}\left(-\dfrac{1}{l+a}+\dfrac{1}{a}\right) = -\dfrac{Mm}{a(a+l)}.$ *Answer.*

CHAPTER V

PARTIAL DERIVATIVES

35. Functions of More than One Variable. In the preceding chapters we have been concerned with functions of one variable; *i.e.* the variable function depended for its value upon the value of a single variable. Such functions do not by any means suffice for the applications of the Calculus. In fact, the student is already familiar with many examples of a variable whose value depends upon those assigned to two or more distinct variables. Thus the area of a rectangle is a function of *two* variables, viz. the two sides; the volume of a gas depends upon both the pressure and the temperature; the volume of a parallelopiped depends upon the *three* edges, etc.

Notation. If the value of a variable u depends upon two variables, x and y, and can be computed when values are assumed for x and y, then we write precisely as in § 3,

$$u = f(x, y). \tag{46}$$

Similarly for a function of three variables,

$$u = \phi(x, y, z), \text{ etc.}$$

36. Partial Differentiation. As in § 12 the important question arising here is how to determine the manner of variation of the function when the variables change in value. But we have greater latitude here than in § 12. For in (46) we can ask ourselves,

first, how does u vary when x alone varies and y remains constant? or

second, how does u change when x remains constant and y varies? or

third, in what manner does u vary when both x and y change *independently* of each other?

Thus let $u = xy$, x and y being respectively the base and altitude of a rectangle; if y remains constant (say $y = b$), u gives the area of all rectangles of a certain altitude b; and if $x =$ a constant, say a, then u represents the area of all rectangles with common base a. But if x and y both vary independently, then we are to consider *all possible* rectangles.

Now the *first* and *second* cases do not differ in the least from § 12, for we really have in the *first*, u a function of x alone, and in the *second*, u a function of y alone. We can therefore form,

first, the increment quotient (§ 13) when x alone varies, and this is

(47) $$\frac{\Delta u}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}; \text{ or}$$

second, the increment quotient when y alone varies, which is

(48) $$\frac{\Delta u}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

For example, in the area of rectangle already used, $u = xy$,

$$\frac{\Delta u}{\Delta x} = \frac{(x + \Delta x)y - xy}{\Delta x} = y, \text{ and } \frac{\Delta u}{\Delta y} \text{ reduces to } x.$$

Finally, we can, as in § 14, find the limits of the functions in the right-hand members of (47) and (48), in (47) when Δx approaches zero, in (48) when Δy approaches

zero. The results are called the *partial* derivatives of u or $f(x, y)$ with respect to x and y respectively, and this step of passing to the limit we indicate on the left by replacing the Δ's by *round* ∂'s, so that

$$(49) \qquad \frac{\partial u}{\partial x} = \text{Limit}\left(\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right)_{\Delta x=0},$$

$$(50) \qquad \frac{\partial u}{\partial y} = \text{Limit}\left(\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right)_{\Delta y=0}.$$

The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are then to be calculated by the rules of Chapter II, the independent variables being respectively x and y.

EXERCISE 18

1. Find the partial derivatives of:

(1) $u = \log_e\left(\frac{y}{x}\right)$. *Ans.* $\frac{\partial u}{\partial x} = -\frac{1}{x}$; $\frac{\partial u}{\partial y} = \frac{1}{y}$.

(2) $u = \arctan\left(\frac{y}{x}\right)$. *Ans.* $\frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2}$; $\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$.

(3) $u = x^y$. *Ans.* $\frac{\partial u}{\partial x} = yx^{y-1}$; $\frac{\partial u}{\partial y} = x^y \log_e x$.

Partial Differentials:

By § 27, (29), the differential of u, when x alone varies is $\frac{\partial u}{\partial x}dx$, and when y alone varies equals $\frac{\partial u}{\partial y}dy$; these are called the *partial differentials* of u.

$$(51) \quad \begin{cases} \dfrac{\partial u}{\partial x}dx = \text{partial differential of } u \text{, when } x \text{ alone varies;} \\[2ex] \dfrac{\partial u}{\partial y}dy = \text{partial differential of } u \text{, when } y \text{ alone varies.} \end{cases}$$

37. Total Differentiation. We have yet to discuss the third case of § 36, viz. required the change in u when x and y *vary independently*. If Δx, Δy, and Δu are the increments of these variables, then from (46) we have

(52) $$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

By adding and subtracting $f(x, y + \Delta y)$ in the right-hand member, (52) becomes

(53) $$\Delta u = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y).$$

Consider now the last two terms,

$$f(x, y + \Delta y) - f(x, y).$$

This is the increment of u or $f(x, y)$ when y alone varies. Hence, by (27), § 27,

(54) $$f(x, y + \Delta y) - f(x, y) = \frac{\partial u}{\partial y}\Delta y + \text{terms in higher powers of } \Delta y.$$

In the same way the first two terms of (53) give us, if we set

$$u' = f(x, y + \Delta y),$$

(55) $$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial u'}{\partial x}\Delta x + \text{terms involving } \overline{\Delta x}^2, \text{ etc.}$$

But also

$$u' = f(x, y + \Delta y) = f(x, y) + \frac{\partial u}{\partial y}\Delta y + \text{terms in } \overline{\Delta y}^2, \text{ etc.},$$

by (26), § 27. Differentiating with respect to x, we find

(56) $$\frac{\partial u'}{\partial x} = \frac{\partial u}{\partial x} + \text{terms in } \Delta y,$$

since $$u = f(x, y).$$

Consequently, from (56), (55), and (54), (53) becomes

(57) $$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \text{terms of higher degree in } \Delta x, \Delta y.$$

Now letting Δx and Δy approach zero, *i.e.* become the infinitesimals dx and dy, then, as in § 27, calling the *principal part* of Δu the *total differential* of u, we have

(58) $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

From (51) and (58), then, we have the theorem:

The total differential of a function of several variables equals the sum of the partial differentials.

Example. In § 28, Example 1, was demonstrated the result

$$d(xy) = x\,dy + y\,dx,$$

which agrees with (58).

EXERCISE 19

Find the total differentials of the following:

(*a*) $u = \log_e\left(\frac{y}{x}\right)$. *Ans.* $du = \frac{y\,dx - x\,dy}{xy}$.

(*b*) $u = \arctan\left(\frac{y}{x}\right)$. *Ans.* $du = \frac{x\,dy - y\,dx}{x^2 + y^2}$.

(*c*) $u = x^y$. *Ans.* $du = x^{y-1}(y\,dx + x\log_e x\,dy)$.

38. Total Derivative. We may in (57) assume that x and y are not independent, but are functions one of the other, say y a function of x. Then u becomes also a function of x alone, and we may therefore form the total derivative $\frac{du}{dx}$.

Dividing (57) by Δx and taking the limit for $\Delta x = 0$, and $\therefore\ \Delta y = 0$, we have the result

$$(59) \qquad \frac{du}{dx} = \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx},$$

a very important formula.

Suppose in the illustration of the rectangle, § 36, we wish the derivative with respect to the base x of the area u of all rectangles whose altitude y is double the base. Then

$$u = xy,\ y = 2x,\ \frac{\partial u}{\partial x} = y,\ \frac{\partial u}{\partial y} = x,\ \frac{dy}{dx} = 2,$$

and (59) gives $$\frac{du}{dx} = y + 2x = 4x.$$

Or, we may substitute for y before differentiation;

i.e. $$u = x \cdot 2x = 2x^2, \therefore \frac{du}{dx} = 4x, \text{ as before.}$$

Equation (59) is especially important as affording a proof of the method given in § 19. For in the example of that article, set

$$u = x^2 - 3xy + 2y^2 - 3;$$

$$\therefore u = 0, \text{ and } \frac{du}{dx} = 0, \text{ or } \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx} = 0;$$

i.e. (60) $$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{2x - 3y}{-3x + 4y} = \frac{2x - 3y}{3x - 4y},$$

the same answer as before. This formula (60) is very useful.

For further study of the Calculus the student is referred to:

G. A. GIBSON, *An Elementary Treatise on the Calculus.* London, 1901.

YOUNG AND LINEBARGER, *The Elements of the Differential and Integral Calculus.* New York, 1900.

MCMAHON AND SNYDER, *Elements of the Differential Calculus.* New York, 1898.

MURRAY. *An Elementary Course in the Integral Calculus.* New York, 1898.

(183)

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